FIXED POINT THEOREMS FOR B-ENRICHED MULTIVALUED NONEXPANSIVE MAPPINGS AND *-B ENRICHED NONEXPANSIVE MAPPINGS

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***Abstract*—The main purpose of this paper is to extend some fixed point results for single valued b-enriched nonexpansive mappings to the case of multivalued mappings.**
The research of fixed points for single valued mappings and multi valued mappings raises several problems: the existence of the fixed point, the uniqueness of the fixed point and the method(s) of accurate or approximate determination of the fixed point.

A natural question raise by this issue is whether turns single valued fixed point theory for nonexpansive mapping into multivalued mapping. There are some challenges:

- **Definition of mapping**
  There are several problems: correct definition of analogue mapping, establish some convergence theorems
- **Identities**
  The metric induced by norm from X could not be use in 2^X, so must be use another notion of distance. One of them is Pompeiu-Hausdorff distance
- **Inference**
  Some theorems and lemas used in single valued theory don't hold for multivalued theory

Browder demonstrated that nonexpansive nonlinear operators in Banach space has a fixed point, but it is not ensured the uniqueness and indeed the Picard iteration do not converge at a fixed point. Browder and Petrishyn provided an iterative method (Krasnoselskii method), which for demicompact mapping converge strong to a fixed point.

In 1974 Lim, Teck-Cheong [5] provided an existence fixed point theorem in a uniformly convex Banach space for nonexpansive mapping.
The problem raised by the turning of single valued nonexpansive mapping into multivalued nonexpansive mapping is whether the Browder theorem is true in multivalued mappings case, using Pompeiu-Hausdorff metric. The answer given by Ko in [7] at this problem by an example is that the Browder theorem not hold for multivalued nonexpansive mapping.
Genaro Lopez, Hong-Kung Xu in [10] and further more authors provided fixed point theorems for multivalued nonexpansive mappings in Hilbert space and Banach space using an additional strong condition $Tp = \{p\}$ for each $p \in F(T)$ (singleton condition for each fixed point of mapping T).
The b-enriched nonexpansive mappings was turned into bb-enriched multivalued nonexpansive mappings in [8] as it follows: Let $(X, \|\|)$ be a liniar normed space. A multi-valued mapping $T : X \longrightarrow CB(X)$ is called b-enriched multivalued nonexpansive if there exist $b \in [0, 1)$

$$H(bx + Tx, by + Ty) \leq (b + 1)\|x - y\|, \forall x, y \in X \qquad (1)$$

Using the strong condition $Tp = \{p\}$ for each $p \in F(T)$ there was proved the fixed point set of T is closed and convex.

I provide strong and weak convergence fixed point theorems based on Krasnoselskii iterative process as it follows.
Let C be a closed and convex subset of a Hilbert space H. Let $T : C \to CB(C)$ be a b-enriched multivalued nonexpansive mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. For any fixed $x_0 \in C$ and an arbitrary $\theta \in (0, 1)$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by:

$$x_{n+1} = (1 - \theta)x_n + \theta y_n,\ n \geq 0, y_n \in Tx_n \qquad (2)$$

using the strong singleton condition for fixed point of b-enriched multivalued nonexpansive mapping.

To avoid the singleton condition $Tp = p$ for all $p \in F(T)$ way, using definition of *-nonexpansive mapping introduced and studied by Hussain and Latif in [6], b-enriched nonexpansive mapping have been turned into *-b enriched nonexpansive mapping. For *-b-enriched nonexpansive mapping we provided some fixed point theorems. A multivalued mapping $T : C \longrightarrow 2^X, C \subset X$ is said *-b-enriched nonexpansive mapping if there exists $b \in [0, \infty)$ such that for all $x, y \in C$ and $u_x \in Tx$ with $d(x, u_x) = d(x, Tx)$ there exist $u_y \in Ty$ with $d(y, u_y) = d(y, Ty)$ such as

$$\|(b(x - y) + (u_x - u_y)\| \leq (b + 1)\|x - y\| \qquad (3)$$

$\forall x, y \in X$. **.**

***Index Terms*—**

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