Fixed Point Theorems in Rectangular b-Metric Space Endowed with a Partial Order

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Introduction

1912 – the year Brouwer's Theorem was published;

1922 – the year Banach's Contraction Mapping Principle was published;

If X is a nonempty set and $T : X \to X$ is a map, then

$$Fix\left(T\right) = \left\{x \in X | x = Tx\right\}.$$

Geometric interpretation

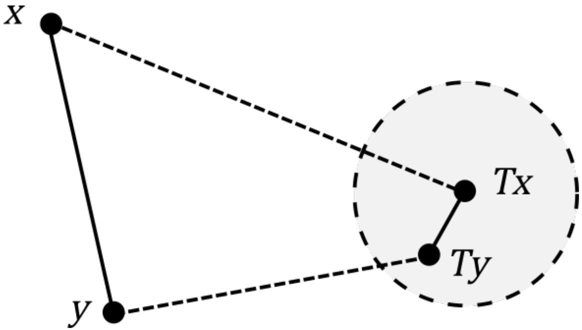

Figure: Geometric interpretation of contractive condition

Geometric interpretation

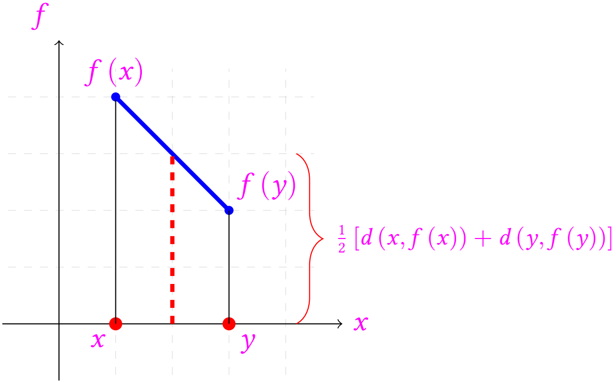

Figure: A geometric interpretation of Kannan contractive condition [6].

Mulțimi parțial ordonate

Example

Let $X = C(a, b, \mathbb{R}_+)$ be the set of all continuous functions from compact subset $[a, b] \subset \mathbb{R}$ to the set of non-negative real numbers. Let $[c, d] \subset [a, b]$ be a compact subset and $\mu \in (0, 1)$, $t' \in [c, d]$ be two positive real numbers. For any $u, v \in X$, if we consider that $u \preccurlyeq v$ if and only if $\mu \cdot v(t) - v(t') \leq \mu \cdot u(t) - u(t')$ for all $t \in [a, b]$, then $(X, \preccurlyeq)$ is a partially ordered set of continuous functions.

FPT in partially ordered set

Theorem ([9])

Let $(X, \preceq)$ be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Let (X, d) be a complete metric space and $f : X \to X$ be a continuous and monotone (i.e., either decreasing or increasing) operator. Suppose that the following two assertions hold:

(H-i.) *There exists $a \in (0,1)$ such that $d(f(x), f(y)) \leq a \cdot d(x, y)$ for each $x, y \in X$, with $x, y \in X_{\preceq}$;*

(H-ii.) *There exists $x_0 \in X$ such that $(x_0, f(x_0)) \in X_{\preceq}$.*

Then, f has an unique fixed point $x^* \in X$, i.e., $f(x^*) = x^*$, and for each $x \in X$ the sequence $\{f^n(x)\}$ of successive approximations of f starting from x converges to $x^* \in X$.

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rectangular b-metric space

Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a map providing the following axioms.

(M-i.) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(M-ii.) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(M-iii.) there is $s \geq 1$ a real number such that

$$d(x, y) \leq s \cdot (d(x, z_1) + d(z_1, z_2) + d(z_2, y)) \quad (1)$$

is satisfied for all $x, y \in X$, $z_1, z_2 \in X \setminus \{x, y\}$, with $x \neq y$ and $z_1 \neq z_2$.

The pair (X, d) is a rectangular b-metric space with coefficient s if axioms *(M-i.)*, *(M-ii.)* and *(M-iii.)* hold.

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Example

Let (X, d) be a complete metric space, $u, v, w \in X$ and $\{u_n\}$ be a convergent sequence such that $u_n \in X \setminus \{u, v, w\}$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} u_n = u$. Denote by σ the set of all elements of the considered sequence $\{u_n\}$, and $P_v = (\sigma \times \{v\}) \cup (\{v\} \times \sigma)$, $P_w = (\sigma \times \{w\}) \cup (\{w\} \times \sigma)$ and $\Sigma = \sigma \cup \{u, v, w\}$. Define a function δ_Σ from $\Sigma \times \Sigma$ into $[0, \infty)$ by

$$\delta_\Sigma(x, y) = \begin{cases} 0, & \text{if } x = y \\ 2 \cdot \kappa, & \text{if } (x, y) \in \sigma \times \sigma \\ \frac{\kappa}{2} \cdot d(u, u_n), & \text{if } (x, y) \in P_v \cup P_w \\ \kappa, & \text{otherwise} \end{cases}$$

where $\kappa > 0$ is positive real constant. Then, (Σ, δ_Σ) is a rectangular b-metric space.

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Rectangular b-Metric Spaces Endowed with a Partial Order

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Definition

Let $(X, \preceq)$ be a partially ordered set and $s \geq 1$ be a given real number. We say that $(X, d, \preceq)$ is a partially ordered rectangular b-metric space if (X, d) is rectangular b-metric space.

Main Result

Theorem

Let $(X, d, \preceq)$ be a partially ordered rectangular b-metric space with coefficient s, with $s \geq 1$ a given real number. Let $f : X \to X$ be an isotone mapping such that

(T-i.) *There exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$;*

(T-ii.) *There exist $\psi \in \mathcal{S}$ and $\varphi \in \mathcal{E}$ and $L \in (0, \infty)$ such that*

$$\psi\left(s^2 \cdot d\left(f(x)\right), f(y)\right) \leq \psi\left(\mathcal{M}^f_{x,y}\right) - \varphi\left(\mathcal{M}^f_{x,y}\right) + L \cdot m^f_{x,y} \quad (2)$$

for all $x, y \in X_{\leq}$;

Then, f has a fixed point in X.

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We make the notations

$$\mathcal{M}^f_{x,y} = \max\left\{d\left(x,y\right), d\left(x,f\left(x\right)\right), d\left(y,f\left(y\right)\right)\right\} \tag{3}$$

and

$$m^f_{x,y} = \min\left\{d\left(x,f\left(x\right)\right), d\left(y,f\left(y\right)\right), d\left(x,f\left(y\right)\right), d\left(y,f\left(x\right)\right)\right\}. \tag{4}$$

If $y = f\left(x\right)$, then

$$\mathcal{M}^f_{x,f(x)} = \max\left\{d\left(x,f\left(x\right)\right), d\left(f\left(x\right), f^2\left(x\right)\right)\right\} \tag{5}$$

and

$$m^f_{x,f(x)} = 0 \text{ for all } x \in X. \tag{6}$$

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Proof - hints

– Let $\{y_n\}$ be the Picard iteration defined by $y_0 = x_0$ and $y_n = f(y_{n-1})$ for all $n \geq 1$.
– $\{d(y_n, y_{n+1})\}$ is a real number sequence which is decreasing and positive. Results

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{7}$$

– $\{y_n\}$ is an Rb-Cauchy sequence in (X, d). By the Rb-completeness of (X, d), this implies that $\{y_n\}$ has a limit point $u \in X$
– $d(u, f(u)) = 0$; that is, $u = f(u)$ and u is a fixed point of f.

Conclusions

In this article, we establish a fixed-point theorem in the setting of complete rectangular b-metric spaces endowed with a partial order. We note that several consequences can be obtained from the main result. As another remark, note that the partial order implies a relaxation of contractiveness. As a continuation of this work we indicate the extension of these results to the case of nonself mappings.

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