Nonuniform h-dichotomy with strong invariant projections for discrete time systems in Banach spaces

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Among the asymptotic behaviors of discrete linear systems an important role is played by the dichotomy property and the notion of (uniform) exponential dichotomy is introduced by Perron for differential equations and by Li for difference equations. In the theory of difference equations, subjects with large applications, studied by Agarwal, Daleckii and Krein, Elaydi and Massera in ([1] [12],[16],[20]) a discrete variant of Perron' s results was given by Ta Li in [19]. Several results about exponential dichotomy were obtained by [10], [22], [18].

One of the main reasons for weakening the assumption of uniform exponential dichotomy is that from the point of view of ergodic theory almost all variational equations in a finite-dimensional space admit a nonuniform exponential dichotomy. On the other hand it is important to treat the case of noninvertible systems because of their interest in applications (e.g., random dynamical systems, generated by random parabolic equations, are not invertible).

Characterizations of the nonuniform exponential dichotomy for discrete linear systems can be found in the works [5], [26], [29], [28] [21], [23] and of uniform exponential dichotomy for discrete linear systems can be found in the works [6], [27], [25], [24].

The present paper treats two concepts of h-dichotomy in the nonuniform case for discrete-time systems in Banach spaces, that is h-dichotomy and weak h-dichotomy. These concepts use strongly invariant types of h-dichotomy projections sequences. The main result of the paper is characterizations of Datko-types for these concepts.

1. Preliminaries

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded operators from X into itself. The norms of both these spaces will be denote by $||\cdot||$. Let $\mathbb{N}$ be the set of all positive intergers and we deonte by Δ and T the follwing sets
$\Delta = \{(m,n) \in \mathbb{N}^2 : m \geq n\} \quad T = \{(m,n,p) \in \mathbb{N}^3 : m \geq n \geq p\}$.
In this paper we consider linear discrete-time systems of the form

$$(\mathcal{A}) \quad x_{n+1} = A_n x_n, \quad n \in \mathbb{N}$$

where (A_n) is a sequence in $\mathcal{B}(X)$.
Then every solution $x = (x_n)$ of system $(\mathcal{A})$ is given by

$$x_m = A_m^n x_n, \quad for\ all\ (m,n) \in \Delta,$$

where

$$A_m^n = \begin{cases} A_{m-1}A_{m-2}...A_n, & m > n \\ I, & m = n \end{cases}$$

Remarca 1

We have the following properties:

(i) $A_{n+1}^{n} = A_n$, for all $n \in \mathbb{N}$

(ii) $A_m^n A_n^p = A_m^p$, for all $(m, n, p) \in T$.

Definiţia 2.1

A nondecreasing sequence (h_n) *on* $[1, \infty)$ *is called growth rate sequence if* $\lim_{n\to\infty} h_n = \infty$.

Definiţia 2.2

A sequence (P_n) *on* $\mathcal{B}(X)$ *is called projections sequence on* X *if*

$$P_n^2 = P_n, \quad \text{for all } n \in \mathbb{N}$$

Remarca 2

If (P_n) is projections sequence on X, then the sequence $Q_n = I - P_n$ is also a projections sequence on X, which is called the *complementary projections sequence of* P_n with $KerQ_m = RangeP_m$ and $RangeQ_m = KerP_m$ and $P_mQ_m = Q_mP_m = 0$ for every $m \in \mathbb{N}$.

Definiţia 2.3

The sequence (P_n) is invariant for the linear system $(\mathcal{A})$, if

$$A_n P_n = P_{n+1} A_n, \quad \text{for all } n \in \mathbb{N}$$

Remarca 3

If (P_n) is invariant for $(\mathcal{A})$ then

$$A_m^n P_n = P_m A_m^n \qquad A_m^n Q_n = Q_m A_m^n,$$

for all $(m, n) \in \Delta$.

Remarca 4

If the sequence of projections (P_n) is invariant for the linear system $(\mathcal{A})$, then we also have that the complementary sequence of projections (Q_n) is invariant for the linear system $(\mathcal{A})$.

Definiţia 2.4

The sequence (P_n) is strongly invariant for the linear system $(\mathcal{A})$ if it is invariant for $(\mathcal{A})$ and the restriction of A_m^n is an isomorphism from Range Q_n to Range Q_m.

Remarca 5

If the sequence of projections (P_n) is a strongly invariant for the system $(\mathcal{A})$, then there exists $B : \Delta \to \mathcal{R}$ with $B(n, m) = B_n^m : RangeQ_m \to RangeQ_n$ isomorphism from $Ker\ P_m$ to $Ker\ P_n$ with $A_m^n B_n^m Q_m = Q_m$ and $B_n^m A_m^n Q_n = Q_n$ for all $(m, n) \in \Delta$.

Remarca 6

If (P_n) is a strongly invariant projections sequence for $(\mathcal{A})$ then there exists (B_n), for all $(m, n) \in \Delta$, is an isomorphism from Range Q_m to Range Q_n with the following properties:

1. $A_m^n B_n^m Q_m = Q_m$
2. $B_n^m A_m^n Q_n = Q_n$
3. $B_n^m Q_m = Q_n B_n^m Q_m$
4. $Q_m = B_m^m Q_m = Q_m B_m^m Q_m$
5. $B_p^m Q_m = B_p^n B_n^m Q_m$

for all $(m, n), (n, p) \in \Delta$.

1.1.Nonuniform h-dichotomy with strongly invariant sequence of projections

Definiţia 2.5

Let (P_n) *be a strong invariant sequence projections for the linear system* $(\mathcal{A})$*. The pair* $(\mathcal{A}, P)$ *is nonuniformly h-dichotomic with respect to the sequence projections P if and only if there are* $N \geq 1,\ \nu > 0,\ \epsilon \geq 0$ *such that the following conditions hold:*

(nhd_1^s) $h_m^\nu ||A_m^n P_n x|| \leq N h_n^\nu h_n^\epsilon ||P_n x||$

(nhd_2^s) $h_m^\nu ||B_n^m Q_m x|| \leq N h_n^\nu h_m^\epsilon ||Q_m x||$

for all $(m, n, x) \in \Delta \times X$.

Remarca 7

Let (P_n) be a strong invariant sequence projections for the linear system $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is *nonuniformly h- dichotomic* with respect to the sequence projections P if and only if there are $N \geq 1,\ \nu > 0,\ \epsilon \geq 0$ such that the following conditions hold:
$(nhd_1^{s'})$ $h_m^\nu||A_m^p P_p x|| \leq N h_n^\nu h_n^\epsilon ||A_n^p P_p x||$
$(nhd_2^{s'})$ $h_n^\nu||B_p^m Q_m x|| \leq N h_p^\nu h_m^\epsilon ||B_n^m Q_m x||$
for all $(m, n, p, x) \in T \times X$.

Remarca 8

1. Talking $\epsilon = 0$, in Definition 2.1, it results the *uniform h- dichotomy* property, denote by $(u.h.d.)$.
2. Talking $h_m = e^m$, for all $m \in \mathbb{N}$ in Definition 2.1, it results the *nonuniform exponential dichotomy* property, denote by $(n.e.d.)^s$.
3. Talking $h_m = m + 1$, for all $m \in \mathbb{N}$ in Definition 2.1, it results the *nonuniform polynomial dichotomy* property, denote by $(n.p.d.)^s$.

Definiţia 2.6

Let (P_n) *be a sequence of projections wich is strongly invariant for the* $(\mathcal{A})$. *If the pair* $(\mathcal{A}, P)$ *is weakly nonuniformly h-dichotomic, then there are* $N \geq 1,\ \nu > 0,\ \epsilon \geq 0$ *such that:*

$(wnhd_1^s)$ $h_m^\nu ||A_m^n P_n|| \leq N h_n^\nu h_n^\epsilon ||P_n||$

$(wnhd_2^s)$ $h_m^\nu ||B_n^m Q_m|| \leq N h_n^\nu h_m^\epsilon ||Q_m||$

for all $(m, n) \in \Delta$.

Remarca 9

Let (P_n) be a sequence of projections wich is strongly invariant for the $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is *weakly nonuniformly h-dichotomic* if and only if there are $N \geq 1,\ \nu > 0, \epsilon \geq 0$ such that:

$(wnhd_1^{s'})$ $h_m^{\nu}||A_m^p P_p|| \leq N h_n^{\nu} h_n^{\epsilon}||A_n^p P_p||$

$(wnhd_2^{s'})$ $h_n^{\nu}||B_p^m Q_m|| \leq N h_p^{\nu} h_m^{\epsilon}||B_n^m Q_m||$

for all $(m, n, p) \in T$.

Remarca 10

If the pair $(\mathcal{A})$ is $n.h.d.$ then it is also $w.n.h.d.$ Indeed, it is sufficient to observe that from supremum for $||x|| \leq 1$ in (nhd_1), respectively in (nhd_2), one obtains $(wnhd_1)$ and $(wnhd_2)$.

Remarca 11

1. Talking $\epsilon = 0$, in Definition 2.2, it results the *weak uniform h-dichotomy* property, denote by $(w.u.h.d.)$.
2. Talking $h_m = e^m$, for all $m \in \mathbb{N}$ in Definition 2.2, it results the *weak nonuniform exponential dichotomy* property, denote by $(w.n.e.d.)^s$.
3. Talking $h_m = m+1$, for all $m \in \mathbb{N}$ in Definition 2.2, it results the *weak nonuniform polynomial dichotomy* property, denote by $(w.n.p.d.)^s$.

1.2.Nonuniform h-growth with strongly invariant sequence of projections

Definiţia 2.7

If the pair $(\mathcal{A}, P)$ has nonuniform h-growth (n.h.g), then there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ such that:

(nhg$_1$) $h_n^{\omega}||A_m^p P_p x|| \leq M h_m^{\omega} h_n^{\delta}||A_n^p P_p x||$

(nhg$_2$) $h_p^{\omega}||B_p^m Q_m x|| \leq M h_m^{\omega} h_m^{\delta}||B_n^m Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Remarca 12

The pair $(\mathcal{A}, P)$ has *nonuniform h-growth* if and only if there are $M \geq 1$, $\omega > 0$ and $\delta \geq 0$ such that:

(nhg_1') $h_n^{\omega}||A_m^n P_n x|| \leq M h_m^{\omega} h_n^{\delta} ||P_n x||$

(nhg_2') $h_n^{\omega}||B_n^m Q_m x|| \leq M h_m^{\omega} h_m^{\delta} ||Q_m x||$

for all $(m, n, x) \in \Delta \times X$.

Remarca 13

The particular cases for the concept of nonuniform h-growth are:

1. If $\delta = 0$, we have uniform h-growth.
2. If $h_m = e^m$, we have nonuniform exponential growth.
3. If $h_m = m + 1$, we have nonuniform polynomial growth.

Definiţia 2.8

If the pair $(\mathcal{A}, P)$ *has weak nonuniform h-growth (w.n.h.g), then there are* $M \geq 1, \omega > 0$ *and* $\delta \geq 0$ *such that:*

(wnhg$_1$) $h_n^{\omega}||A_m^p P_p|| \leq M h_m^{\omega} h_n^{\delta}||A_n^p P_p x||$

(wnhg$_2$) $h_p^{\omega}||B_p^m Q_m|| \leq M h_m^{\omega} h_m^{\delta}||B_n^m Q_m||$

for all $(m, n, p) \in T$.

Remarca 14

The pair $(\mathcal{A}, P)$ has *weak nonuniform h-growth* if and only if there are $M \geq 1$, $\omega > 0$ and $\delta \geq 0$ such that:

$(wnhg_1^{'})$ $h_n^{\omega}||A_m^n P_n|| \leq M h_m^{\omega} h_n^{\delta} ||P_n||$

$(wnhg_2^{'})$ $h_n^{\omega}||B_n^m Q_m|| \leq M h_m^{\omega} h_m^{\delta} ||Q_m||$

for all $(m, n) \in \Delta$.

Remarca 15

The particular cases for the concept of weak nonuniform h-growth are:
1. If $\delta = 0$, we have weak uniform h-growth.
2. If $h_m = e^m$, we have weak nonuniform exponential growth.
3. If $h_m = m + 1$, we have weak nonuniform polynomial growth.

2. The main results

In this paper, we will consider $\mathcal{H}$ the set of growth rates (h_n) that satisfy the following properties:

(1) $\exists\ H > 1 : h_{n+1} \leq Hh_n, \forall n \in \mathbb{N}$

(2) $\forall \alpha \in (-1, 0),\ \exists\ H_1 > 1 : \sum_{k=m}^{\infty} h_k^{\alpha} \leq H_1 h_m^{\alpha},\ \forall m \in \mathbb{N}$

(3) $\forall \alpha \in (0, 1),\ \exists\ H_2 > 1 : \sum_{j=0}^{m} h_j^{\alpha} \leq H_2 h_m^{\alpha},\ \forall m \in \mathbb{N}.$

We consider (P_n) a sequence of projections strongly invariant for $(\mathcal{A})$ and (Q_n) the complementary projections sequence of (P_n).

Teorema 3.1

If $(\mathcal{A}, P)$ has nonuniform h-growth and $h \in \mathcal{H}$, the pair $(\mathcal{A}, P)$ is nonuniformly h-dichotomic if and only if exists $D > 1,\ d > 0$ with

$$(nhD_1^s)\ \sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq D h_n^{d+\epsilon} ||A_n^p P_p x||$$

for all $(m, n, p, x) \in T \times X$.

$$(nhD_2^s)\ \sum_{k=p}^{\infty} \frac{h_k^d}{||B_k^m Q_m x||} \leq \frac{D h_p^d h_m^\epsilon}{||B_p^m Q_m x||},$$

for all $B_p^m Q_m x \neq 0$ and $(m, n, p, x) \in T \times X$.

Teorema 3.2

If $(\mathcal{A}, P)$ has nonuniform h-growth, $h \in \mathcal{H}$, then the pair $(\mathcal{A}, P)$ is nonuniformly h-dichotomic if and only if exists $D > 1,\ d > 0$ with:

$$(nhD_2^{s_1}) \quad \sum_{j=n}^{m} \frac{h_j^{-d}}{||A_j^n P_n x||} \leq \frac{D h_n^{-d+\epsilon}}{||A_m^n P_n x||},$$

for all $(m, n, x) \in \Delta \times X$ and $A_m^n P_n x \neq 0$.

$$(nhD_2^{s_2}) \quad \sum_{j=n}^{m} \frac{||B_j^m Q_m x||}{h_j^d} \leq \frac{D||Q_m x||}{h_m^{d+\epsilon}},$$

for all $(m, n, p, x) \in T \times X$.

Teorema 3.3

If $(\mathcal{A}, P)$ *has* $h \in \mathcal{H}$*, the pair* $(\mathcal{A}, P)$ *is weakly nonuniformly h-dichotomic if and only if exists* $D > 1,\ d > 0$ *with*

$$(wnhD_1^s)\ \sum_{k=n}^{\infty} h_k^d ||A_k^p P_p|| \leq D h_n^{d+\epsilon} ||A_n^p P_p||$$

for all $(m, n, p) \in T$.

$$(wnhD_2^s)\ \sum_{k=p}^{\infty} \frac{h_k^d}{||B_k^m Q_m||} \leq \frac{D h_p^d h_m^\epsilon}{||B_p^m Q_m||},$$

for all $B_p^m Q_m x \neq 0$ *and* $(m, n, p) \in T$.

Teorema 3.4

If $(\mathcal{A}, P)$ *has* $h \in \mathcal{H}$, *then the pair* $(\mathcal{A}, P)$ *is weakly nonuniformly h-dichotomic if and only if exists* $D > 1,\ d > 0$ *with:*

$$(wnhD_2^{s_1})\ \ \sum_{j=n}^{m} \frac{h_j^{-d}}{||A_j^n P_n||} \leq \frac{Dh_n^{-d+\epsilon}}{||A_m^n P_n||},$$

for all $(m, n) \in \Delta$ *and* $A_m^n P_n \neq 0$.

$$(wnhD_2^{s_2})\ \ \sum_{j=n}^{m} \frac{||B_j^m Q_m||}{h_j^d} \leq \frac{D||Q_m||}{h_m^{d+\epsilon}},$$

for all $(m, n, p) \in T$.

Open problems: The theorems of Barbashin type for uniform and nonuniform cases with growth rates for discrete time systems in Banach spaces.

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Thank you for attention!