Nonuniform h-dichotomy with strong invariant projections for discrete time systems in Banach spaces

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Abstract

The present paper treats two concepts of h-dichotomy in the nonuniform case for discrete-time systems in Banach spaces, that is h-dichotomy and weak h-dichotomy. These concepts use strongly invariant types of h-dichotomy projections sequences. The main result of the paper is characterizations of Datko-types for these concepts.

Keywords: *discrete time systems, growth rate, strong invariant projections, nonuniform h-dichotomy, weak nonuniform h-dichotomy.*

1. Introduction

Among the asymptotic behaviors of discrete linear systems an important role is played by the dichotomy property and the notion of (uniform) exponential dichotomy is introduced by Perron for differential equations and by Li for difference equations. In the theory of difference equations, subjects with large applications, ([1] [12], [16], [20]). A discrete variant of Perron's results was given by Ta Li in [19]. Several results about exponential dichotomy were obtained by [10], [22], [18]. In some situations, particularly in the nonautonomous settings, the concept of uniform exponential dichotomy is too restrictive and it is important to consider more general behaviors. One of the main reasons for weakening the assumption of uniform exponential dichotomy is that from the point of view of ergodic theory almost all variational equations in a finite-dimensional space admit a nonuniform exponential dichotomy. On the other hand it is important to treat the case of noninvertible systems because of their interest in applications (e.g., random dynamical systems, generated by random parabolic equations, are not invertible).

Characterizations of the nonuniform exponential dichotomy for discrete linear systems can be found in the works [5], [26], [29], [28] [21], [23] and of uniform exponential dichotomy for discrete linear systems can be found in the works [6], [27], [25], [24].

This paper presents two concepts of nonuniform h-dichotomy (h-dichotomy, weak h-dichotomy) with strong invariant projections for discrete-time systems in Banach spaces. In addition, in this paper we obtain different characterizations of Datko-type for these concepts.

2. Preliminaries

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded operators from X into itself. The norms of both these spaces will be denote by $||\cdot||$. Let $\mathbb{N}$ be the set of all positive intergers and we deonte by Δ and T the follwing sets

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$\Delta = \{(m,n) \in \mathbb{N}^2 : m \geq n\} \quad T = \{(m,n,p) \in \mathbb{N}^3 : m \geq n \geq p\}$.
In this paper we consider linear discrete-time systems of the form

$$(\mathcal{A}) \qquad x_{n+1} = A_n x_n, \quad n \in \mathbb{N}$$

where (A_n) is a sequence in $\mathcal{B}(X)$.
Then every solution $x = (x_n)$ of system $(\mathcal{A})$ is given by

$$x_m = A_m^n x_n, \quad for\ all\ (m,n) \in \Delta,$$

where

$$A_m^n = \begin{cases} A_{m-1}A_{m-2}...A_n, & m > n \\ I, & m = n \end{cases}$$

and I is the identity operator on X.

Remark 2.1. *We have the following properties:*
(i) $A_{n+1}^n = A_n$, *for all* $n \in \mathbb{N}$
(ii) $A_m^n A_n^p = A_m^p$, *for all* $(m,n,p) \in T$.

Definition 2.1. *A nondecreasing sequence* (h_n) *on* $[1,\infty)$ *is called growth rate sequence if* $\lim_{n\to\infty} h_n = \infty$.

Definition 2.2. *A sequence* (P_n) *on* $\mathcal{B}(X)$ *is called projections sequence on* X *if*

$$P_n^2 = P_n, \quad for\ all\ n \in \mathbb{N}$$

Remark 2.2. *If* (P_n) *is projections sequence on X, then the sequence* $Q_n = I - P_n$ *is also a projections sequence on X, which is called the complementary projections sequence of* P_n *with* $KerQ_m = RangeP_m$ *and* $RangeQ_m = KerP_m$ *and* $P_mQ_m = Q_mP_m = 0$ *for every* $m \in \mathbb{N}$.

Definition 2.3. *The sequence* (P_n) *is invariant for the linear system* $(\mathcal{A})$, *if*

$$A_nP_n = P_{n+1}A_n, \quad for\ all\ n \in \mathbb{N}$$

Remark 2.3. *If* (P_n) *is invariant for* $(\mathcal{A})$ *then*

$$A_m^n P_n = P_m A_m^n \qquad\qquad A_m^n Q_n = Q_m A_m^n,$$

for all $(m,n) \in \Delta$.

Proof. It is immediate. □

Remark 2.4. *If the sequence of projections* (P_n) *is invariant for the linear system* $(\mathcal{A})$, *then we also have that the complementary sequence of projections* (Q_n) *is invariant for the linear system* $(\mathcal{A})$.

Definition 2.4. *The sequence* (P_n) *is strongly invariant for the linear system* $(\mathcal{A})$ *if it is invariant for* $(\mathcal{A})$ *and the restriction of* A_m^n *is an isomorphism from Range* Q_n *to Range* Q_m.

Remark 2.5. *If the sequence of projections* (P_n) *is a strongly invariant for the system* $(\mathcal{A})$, *then there exists* $B : \Delta \to \mathcal{R}$ *with* $B(n,m) = B_n^m : RangeQ_m \to RangeQ_n$ *isomorphism from* $Ker\ P_m$ *to* $Ker\ P_n$ *with* $A_m^n B_n^m Q_m = Q_m$ *and* $B_n^m A_m^n Q_n = Q_n$ *for all* $(m,n) \in \Delta$.

Remark 2.6. *If* (P_n) *is a strongly invariant projections sequence for* $(\mathcal{A})$ *then there exists* (B_n), *for all* $(m,n) \in \Delta$, *is an isomorphism from Range* Q_m *to Range* Q_n *with the following properties:*

1. $A_m^n B_n^m Q_m = Q_m$
2. $B_n^m A_m^n Q_n = Q_n$
3. $B_n^m Q_m = Q_n B_n^m Q_m$
4. $Q_m = B_m^m Q_m = Q_m B_m^m Q_m$
5. $B_p^m Q_m = B_p^n B_n^m Q_m$

for all $(m,n), (n,p) \in \Delta$.

Proof. See [5]. □

2.1. Nonuniform h-dichotomy with strongly invariant sequence of projections

Definition 2.5. *Let (P_n) be a strong invariant sequence projections for the linear system $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is nonuniformly h-dichotomic with respect to the sequence projections P if and only if there are $N \geq 1,\ \nu > 0,\ \epsilon \geq 0$ such that the following conditions hold:*

(nhd_1^s) $h_m^\nu ||A_m^n P_n x|| \leq N h_n^\nu h_n^\epsilon ||P_n x||$

(nhd_2^s) $h_m^\nu ||B_n^m Q_m x|| \leq N h_n^\nu h_m^\epsilon ||Q_m x||$

for all $(m, n, x) \in \Delta \times X$.

Remark 2.7. *Let (P_n) be a strong invariant sequence projections for the linear system $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is nonuniformly h- dichotomic with respect to the sequence projections P if and only if there are $N \geq 1,\ \nu > 0,\ \epsilon \geq 0$ such that the following conditions hold:*

$(nhd_1^{s'})$ $h_m^\nu ||A_m^p P_p x|| \leq N h_n^\nu h_n^\epsilon ||A_n^p P_p x||$

$(nhd_2^{s'})$ $h_n^\nu ||B_p^m Q_m x|| \leq N h_p^\nu h_m^\epsilon ||B_n^m Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Proof. Necessity. If the pair $(\mathcal{A}, P)$ is n.h.d. with the respect to the sequence of projections P then

$$h_m^\nu ||A_m^p P_p x|| = h_m^\nu ||A_m^n P_n A_n^p P_p x|| \leq N h_n^\nu h_n^\epsilon ||A_n^p P_p x||$$

and

$$h_n^\nu ||B_p^m Q_m x|| = ||B_p^n B_n^m Q_m x|| =$$
$$= ||B_p^n Q_n B_n^m Q_m x|| \leq N h_p^\nu h_m^\epsilon ||Q_n B_n^m Q_m x|| =$$
$$= N h_p^\nu h_m^\epsilon ||B_n^m Q_m x||$$

for all $(m, n, p, x) \in T \times X$.

Sufficiency. Immediatly, for $p = n$ in $(nhd_1^{s'})$ and $p = n$ in $(nhd_2^{s'})$.

□

Remark 2.8. *1. Talking $\epsilon = 0$, in Definition 2.5, it results the uniform h- dichotomy property, denote by (u.h.d.).*

2. Talking $h_m = e^m$, for all $m \in \mathbb{N}$ in Definition 2.5, it results the nonuniform exponential dichotomy property, denote by $(n.e.d.)^s$.

3. Talking $h_m = m + 1$, for all $m \in \mathbb{N}$ in Definition 2.5, it results the nonuniform polynomial dichotomy property, denote by $(n.p.d.)^s$.

Definition 2.6. *Let (P_n) be a sequence of projections wich is strongly invariant for the $(\mathcal{A})$. If the pair $(\mathcal{A}, P)$ is weakly nonuniformly h-dichotomic, then there are $N \geq 1,\ \nu > 0,\ \epsilon \geq 0$ such that:*

$(wnhd_1^s)$ $h_m^\nu ||A_m^n P_n|| \leq N h_n^\nu h_n^\epsilon ||P_n||$

$(wnhd_2^s)$ $h_m^\nu ||B_n^m Q_m|| \leq N h_n^\nu h_m^\epsilon ||Q_m||$

for all $(m, n) \in \Delta$.

Remark 2.9. *Let (P_n) be a sequence of projections wich is strongly invariant for the $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is weakly nonuniformly h-dichotomic if and only if there are $N \geq 1,\ \nu > 0, \epsilon \geq 0$ such that:*

$(wnhd_1^{s'})$ $h_m^\nu ||A_m^p P_p|| \leq N h_n^\nu h_n^\epsilon ||A_n^p P_p||$

$(wnhd_2^{s'})$ $h_n^\nu ||B_p^m Q_m|| \leq N h_p^\nu h_m^\epsilon ||B_n^m Q_m||$

for all $(m, n, p) \in T$.

Proof. Necessity. If the pair $(\mathcal{A}, P)$ is w.n.h.d. with the respect to the sequence of projections P then

$$h_m^\nu ||A_m^p P_p|| = h_m^\nu ||A_m^n P_n A_n^p P_p|| \leq N h_n^\nu h_n^\epsilon ||A_n^p P_p||$$

and

$$h_n^\nu ||B_p^m Q_m|| = ||B_p^n B_n^m Q_m|| =$$
$$= ||B_p^n Q_n B_n^m Q_m|| \leq N h_p^\nu h_m^\epsilon ||Q_n B_n^m Q_m|| =$$

$$= Nh_p^{\nu}h_m^{\epsilon}||B_n^m Q_m||$$

for all $(m,n,p) \in T \times X$.

Sufficiency. Immediatly, for $p = n$ in $(wnhd_1^{s'})$ and $p = n$ in $(wnhd_2^{s'})$. □

Remark 2.10. *1. Talking $\epsilon = 0$, in Definition 2.6, it results the weaak uniform h- dichotomy property, denote by (w.u.h.d.).*

2. Talking $h_m = e^m$, for all $m \in \mathbb{N}$ in Definition 2.6, it results the weak nonuniform exponential dichotomy property, denote by $(w.n.e.d.)^s$.

3. Talking $h_m = m + 1$, for all $m \in \mathbb{N}$ in Definition 2.6, it results the weak nonuniform polynomial dichotomy property, denote by $(w.n.p.d.)^s$.

Remark 2.11. *If the pair $(\mathcal{A})$ is n.h.d. then it is also w.n.h.d. Indeed, it is sufficient to observe that from supremum for $||x|| \leq 1$ in (nhd_1), respectively in (nhd_2), one obtains $(wnhd_1)$ and $(wnhd_2)$.*

2.2. Nonuniform h-growth with strongly invariant sequence of projections

Definition 2.7. *If the pair $(\mathcal{A}, P)$ has nouniform h-growth (n.h.g), then there are $M \geq 1, \omega > 0$ andd $\delta \geq 0$ such that:*

(nhg_1) $h_n^{\omega}||A_m^p P_p x|| \leq Mh_m^{\omega}h_n^{\delta}||A_n^p P_p x||$

(nhg_2) $h_p^{\omega}||B_p^m Q_m x|| \leq Mh_m^{\omega}h_m^{\delta}||B_n^m Q_m x||$

for all $(m,n,p,x) \in T \times X$.

Remark 2.12. *The pair $(\mathcal{A}, P)$ has nonuniform h-growth if and only if there are $M \geq 1$, $\omega > 0$ and $\delta \geq 0$ such that:*

$(nhg_1^{'})$ $h_n^{\omega}||A_m^n P_n x|| \leq Mh_m^{\omega}h_n^{\delta}||P_n x||$

$(nhg_2^{'})$ $h_n^{\omega}||B_n^m Q_m x|| \leq Mh_m^{\omega}h_m^{\delta}||Q_m x||$

for all $(m,n,x) \in \Delta \times X$.

Proof. Necessity: It is obvious for $p = n$ in (nhg_1) and $p = n, \quad n = m$ in (nhg_2).

Sufficiency: For $(nhg_1') \implies (nhg_1)$ we have

$$h_n^{\omega}||A_m^p P_p x|| = h_n^{\omega}||A_m^n A_n^p P_p x|| \leq$$

$$\leq Mh_m^{\omega}h_n^{\delta}||A_n^p P_p x||$$

and for $(nhg_2') \implies (nhg_2)$ we have for $n = p$ in (nhg_2').

□

Remark 2.13. *The particular cases for the concept of nonuniform h-growth are: 1. If $\delta = 0$, we have nonuniform h-dichotomy.*

2. If $h_m = e^m$, we have nonuniform exponential growth.

3. If $h_m = m + 1$, we have nonuniform polynomial growth.

Definition 2.8. *If the pair $(\mathcal{A}, P)$ has weak nonuniform h-growth (w.n.h.g), then there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ such that:*

$(wnhg_1)$ $h_n^{\omega}||A_m^p P_p|| \leq Mh_m^{\omega}h_n^{\delta}||A_n^p P_p x||$

$(wnhg_2)$ $h_p^{\omega}||B_p^m Q_m|| \leq Mh_m^{\omega}h_m^{\delta}||B_n^m Q_m||$

for all $(m,n,p) \in T$.

Remark 2.14. *The pair $(\mathcal{A}, P)$ has weak nonuniform h-growth if and only if there are $M \geq 1$, $\omega > 0$ and $\delta \geq 0$ such that:*

$(wnhg_1^{'})$ $h_n^{\omega}||A_m^n P_n|| \leq Mh_m^{\omega}h_n^{\delta}||P_n||$

$(wnhg_2^{'})$ $h_n^{\omega}||B_n^m Q_m|| \leq Mh_m^{\omega}h_m^{\delta}||Q_m||$

for all $(m,n) \in \Delta$.

Proof. *Necessity:* It is obvious for $p = n$ in $(wnhg_1)$ and $(wnhg_2)$.

Sufficiency: For $(wnhg_1') \implies (wnhg_1)$ we have

$$h_n^{\omega}||A_m^p P_p|| = h_n^{\omega}||A_m^n A_n^p P_p|| \leq$$

$$\leq M h_m^{\omega} h_n^{\delta}||A_n^p P_p||$$

and for $(wnhg_2') \implies (wnhg_2)$ we have for $n = p$ in $(wnhg_2')$.

□

Remark 2.15. *The particular cases for the concept of weak nonuniform h-growth are: 1. If $\delta = 0$, we have weak nonuniform h-dichotomy.*

2. If $h_m = e^m$, we have weak nonuniform exponential growth.

3. If $h_m = m + 1$, we have weak nonuniform polynomial growth.

3. The main results

In this paper, we will consider $\mathcal{H}$ the set of growth rates (h_n) that satisfy the following properties:

(1) $\exists\ H > 1 : h_{n+1} \leq H h_n, \forall n \in \mathbb{N}$

(2) $\forall \alpha \in (-1, 0),\ \exists\ H_1 > 1 : \sum_{k=m}^{\infty} h_k^{\alpha} \leq H_1 h_m^{\alpha},\ \forall m \in \mathbb{N}$

(3) $\forall \alpha \in (0, 1),\ \exists\ H_2 > 1 : \sum_{j=0}^{m} h_j^{\alpha} \leq H_2 h_m^{\alpha},\ \forall m \in \mathbb{N}.$

We consider (P_n) a sequence of projections strongly invariant for $(\mathcal{A})$ and (Q_n) the complementary projections sequence of (P_n).

Theorem 3.1. *If $(\mathcal{A}, P)$ has nonuniform h-growth and $h \in \mathcal{H}$, the pair $(\mathcal{A}, P)$ is nonuniformly h-dichotomic if and only if exists $D > 1,\ d > 0$ with*

(nhD_1^s) $\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq D h_n^{d+\epsilon} ||A_n^p P_p x||$

for all $(m, n, p, x) \in T \times X$.

(nhD_2^s) $\sum_{k=p}^{\infty} \frac{h_k^d}{||B_k^m Q_m x||} \leq \frac{D h_p^d h_m^{\epsilon}}{||B_p^m Q_m x||},$

for all $B_p^m Q_m x \neq 0$ and $(m, n, p, x) \in T \times X$.

Proof. *Necessity* Let be $d \in (0, \nu)$

(nhD_1^s)

$$\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq \sum_{k=n}^{\infty} h_k^d h_k^{\epsilon} N \left(\frac{h_k}{h_n}\right)^{-\nu} ||A_n^p P_p x|| \leq$$

$$\leq N h_n^{\nu} ||A_n^p P_p x|| \sum_{k=n}^{\infty} h_k^{d-\nu+\epsilon} \leq N H_1 h_n^{d+\epsilon} ||A_n^p P_p x|| =$$

$$= D h_n^{d+\epsilon} ||A_n^p P_p x||$$

where $D = N H_1 > 1$ and $(n, p, x) \in \Delta \times X$.

(nhD_2^s)

$$\sum_{k=p}^{\infty} \frac{h_k^d}{||B_k^m Q_m x||} \leq \sum_{k=p}^{\infty} N h_k^d h_k^{\epsilon} \left(\frac{h_p}{h_k}\right)^{\nu} \cdot \frac{1}{||B_p^m Q_m x||} \leq$$

$$\leq \frac{N h_p^{\nu}}{||B_p^m Q_m x||} \sum_{k=p}^{\infty} h_k^{d+\epsilon-\nu} \leq \frac{N H_1 h_p^{d+\epsilon}}{||B_p^m Q_m x||} \leq \frac{D h_p^d h_m^{\epsilon}}{||B_p^m Q_m x||}$$

where $D = NH_1 > 1$ for all $B_p^m Q_m x \neq 0$ and $(m,n,p,x) \in T \times X$.

For the sufficiency, if we consider $k = m$ in (nhD_1^s), respectively $k = n$ in (nhD_2^s), we obtain (nhd_1) and (nhd_2).

□

Theorem 3.2. *If $(\mathcal{A}, P)$ has nonuniform h-growth, $h \in \mathcal{H}$, then the pair $(\mathcal{A}, P)$ is nonuniformly h-dichotomic if and only if exists $D > 1,\ d > 0$ with:*

$(nhD_2^{s_1})$ $\displaystyle\sum_{j=n}^{m} \frac{h_j^{-d}}{||A_j^n P_n x||} \leq \frac{D h_n^{-d+\epsilon}}{||A_m^n P_n x||},$

for all $(m,n,x) \in \Delta \times X$ and $A_m^n P_n x \neq 0$.

$(nhD_2^{s_2})$ $\displaystyle\sum_{j=n}^{m} \frac{||B_j^m Q_m x||}{h_j^d} \leq \frac{D||Q_m x||}{h_m^{d+\epsilon}},$

for all $(m,n,p,x) \in T \times X$.

Proof. Necessity. Let be $d \in (0, \nu)$.

$(nhD_2^{s_1})$

$$\sum_{j=n}^{m} \frac{h_j^{-d}}{||A_j^n P_n x||} \leq \sum_{j=n}^{m} N h_j^{-d} \cdot \left(\frac{h_j}{h_m}\right)^{\nu} h_j^{\epsilon} \cdot \frac{1}{||A_m^n P_n x||} \leq$$

$$\leq \frac{N h_m^{-\nu}}{||A_m^n P_n x||} \cdot \sum_{j=p}^{m} h_j^{\nu-d+\epsilon} \leq \frac{N H_2 h_n^{-d+\epsilon}}{||A_m^n P_p n x||} =$$

$$= \frac{D h_n^{-d+\epsilon}}{||A_m^n P_n x||}$$

where $D = NH_2 > 1$ and $A_m^n P_n x \neq 0$ for all $(m,n,p,x) \in T \times X$.

$$\sum_{j=n}^{m} \frac{||B_j^m Q_m x||}{h_j^d} \leq$$

$$\leq \sum_{j=p}^{m} N h_j^{-d} \left(\frac{h_m}{h_j}\right)^{-\nu} h_m^{\epsilon} ||A_m^j Q_j B_j^m Q_m x|| \leq$$

$$\leq N h_m^{-\nu+\epsilon} ||Q_m x|| \sum_{j=n}^{m} h_j^{\nu-d} \leq D h_m^{-d+\epsilon} ||Q_m x||$$

where $D = NH_2 > 1$.

Sufficiency. For $j = n$ in $(nhD_1^{s_1})$ and $(nhD_2^{s_2})$ we obtain (nhd_1^s) and (nhd_2^s).

□

Theorem 3.3. *If $(\mathcal{A}, P)$ has $h \in \mathcal{H}$, the pair $(\mathcal{A}, P)$ is weakly nonuniformly h-dichotomic if and only if exists $D > 1,\ d > 0$ with*

$(wnhD_1^s)$ $\displaystyle\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p|| \leq D h_n^{d+\epsilon} ||A_n^p P_p||$

for all $(m,n,p) \in T$.

$(wnhD_2^s)$ $\displaystyle\sum_{k=p}^{\infty} \frac{h_k^d}{||B_k^m Q_m||} \leq \frac{D h_p^d h_m^{\epsilon}}{||B_p^m Q_m||},$

for all $B_p^m Q_m x \neq 0$ and $(m,n,p) \in T$.

Proof. *Necessity*

Let be $d \in (0, \nu)$

$(wnhD_1^s)$

$$\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p|| \leq \sum_{k=n}^{\infty} h_k^d h_k^{\epsilon} N \left(\frac{h_k}{h_n}\right)^{-\nu} ||A_n^p P_p|| \leq$$

$$\leq N h_n^{\nu} ||A_n^p P_p|| \sum_{k=n}^{\infty} h_k^{d-\nu+\epsilon} \leq N H_1 h_n^{d+\epsilon} ||A_n^p P_p|| =$$

$$= D h_n^{d+\epsilon} ||A_n^p P_p||$$

where $D = NH_1 > 1$ and $(n, p) \in \Delta$.

$(wnhD_2^s)$

$$\sum_{k=p}^{\infty} \frac{h_k^d}{||B_k^m Q_m||} \leq \sum_{k=p}^{\infty} N h_k^d h_k^{\epsilon} \left(\frac{h_p}{h_k}\right)^{\nu} \cdot \frac{1}{||B_p^m Q_m||} \leq$$

$$\leq \frac{N h_p^{\nu}}{||B_p^m Q_m||} \sum_{k=p}^{\infty} h_k^{d+\epsilon-\nu} \leq \frac{N H_1 h_p^{d+\epsilon}}{||B_p^m Q_m||} \leq \frac{D h_p^d h_m^{\epsilon}}{||B_p^m Q_m||}$$

where $D = NH_1 > 1$ for all $B_p^m Q_m \neq 0$ and $(m, n, p) \in T$.

For the sufficiency, if we consider $k = m$ in $(wnhD_1^s)$ and $k = n$ in $(wnhD_2^s)$, we obtain $(wnhd_1^s)$ and $(wnhd_2^s)$.

□

Theorem 3.4. *If $(\mathcal{A}, P)$ has $h \in \mathcal{H}$, then the pair $(\mathcal{A}, P)$ is weakly nonuniformly h-dichotomic if and only if exists $D > 1,\ d > 0$ with:*

$(wnhD_2^{s_1})$ $\displaystyle\sum_{j=n}^{m} \frac{h_j^{-d}}{||A_j^n P_n||} \leq \frac{D h_n^{-d+\epsilon}}{||A_m^n P_n||}$,

for all $(m, n) \in \Delta$ and $A_m^n P_n \neq 0$.

$(wnhD_2^{s_2})$ $\displaystyle\sum_{j=n}^{m} \frac{||B_j^m Q_m||}{h_j^d} \leq \frac{D ||Q_m||}{h_m^{d+\epsilon}}$,

for all $(m, n, p) \in T$.

Proof. *Necessity.* Let be $d \in (0, \nu)$.

$(wnhD_2^{s_1})$

$$\sum_{j=n}^{m} \frac{h_j^{-d}}{||A_j^n P_n||} \leq \sum_{j=n}^{m} N h_j^{-d} \cdot \left(\frac{h_j}{h_m}\right)^{\nu} h_j^{\epsilon} \cdot \frac{1}{||A_m^n P_n||} \leq$$

$$\leq \frac{N h_m^{-\nu}}{||A_m^n P_n||} \cdot \sum_{j=p}^{m} h_j^{\nu-d+\epsilon} \leq \frac{N H_2 h_n^{-d+\epsilon}}{||A_m^n P_n||} =$$

$$= \frac{D h_n^{-d+\epsilon}}{||A_m^n P_n||}$$

where $D = NH_2 > 1$ and $A_m^n P_n \neq 0$ for all $(m, n, p) \in T$.

$$\sum_{j=n}^{m} \frac{||B_j^m Q_m||}{h_j^d} \leq$$

$$\leq \sum_{j=p}^{m} N h_j^{-d} \left(\frac{h_m}{h_j}\right)^{-\nu} h_m^{\epsilon} ||A_m^j Q_j B_j^m Q_m|| \leq$$

$$\leq N h_m^{-\nu+\epsilon}||Q_m||\sum_{j=n}^{m} h_j^{\nu-d} \leq D h_m^{-d+\epsilon}||Q_m||$$

where $D = NH_2 > 1$.

Sufficiency. For $j = n$ in $(wnhD_1^{s_1})$ and $(wnhD_2^{s_2})$ we obtain $(wnhd_1^s)$ and $(wnhd_2^s)$.

□

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