Integral characterizations for uniform dichotomy in mean with growth rates for reversible stochastic skew-evolution semiflows in Banach spaces

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Outline of this talk

The content of the presentation

Introduction

Introduction

In recent decades within mathematical literature, one of the most important topics discussed in the field of dynamical systems is the uniform exponential behaviour. This concept was introduced by O. Perron in (see [19]), while studying the connection between the conditional stability of an equation $\dot{x}(t) = A(t)x$ and the existence of bounded solutions of the equation $\dot{x}(t) = A(t)x + f(t,x)$.

The property of exponential dichotomy for linear differential equations has gained importance since the appearance of two fundamental monographs due to J. L. Massera & J. J. Schäffer in 1966 [12] and J. L. Daleckii & M.G. Krein in 1974 [7]. Since then, there have been a number of works devoted to this problem, such as [5], [8], and [11].

Introduction

The asymptotic behaviour of stochastic evolution equations in infinite dimensional spaces has proved to be a research area of large intensity. Based on the stochastic equations studied in monographs by L. Arnold [1] and D. Prato and J. Zabczyk [20] were born important examples of stochastic evolution semiflows.

The exponential dichotomy in a stochastic setting was discused by many authors, such as A. M. Ateiwi in [2] or T. Caraballo et al. in [6].

Introduction

The notion of skew-evolution semiflow became a front-line topic in the modern theory of dynamical system and differential equations. In the deterministic setting it can be traced back to the works of M. Megan and C. Stoica in [14] and generalizes the notion of: evolution operators, semigroups of operators and skew-product semiflows (see [1, 8, 15, 13, 17, 16, 20]). The property of dichotomy for stochastic skew-evolution semiflows in Banach spaces is treated in [10], [22, 23, 24, 25].

Throughout the years an important extension of exponential dichotomy and polynomial dichotomy is introduced by Pinto in his work in 1984 [18] with the intention of obtaining results about stability for a weakly stable system under some perturbations. This concept is called dichotomy with growth rates or h-dichotomy, where by the growth rate, we understand a bijective and nondecreasing application $h : \mathbb{R}_+ \to [1, \infty)$ with $\lim_{t\to\infty} h(t) = \infty$.

Introduction

Datko's theorem was the starting point for important studies concerning the uniform exponential stability of evolution equations. After the seminal research of Datko [9], there has been a large number of papers devoted to this subject. Generalizations of Datko's results for the case of the polynomial behaviors are given in [3, 4].

In the present paper, we approach the concept of uniform dichotomy in mean with growth rates and the major result is the characterization of Datko type for the uniform h-dichotomy in mean regarding invariant projections families to the reversible stochastic skew-evolution semiflows. As particular cases of the notion studied in our paper, we obtain the concept of uniform exponential dichotomy in mean and uniform polynomial dichotomy in mean respectively.

At the West University of Timişoara there is a Seminar of Scientific Research initiated by Prof. Mircea Reghiş and continued by Prof. Univ. Emerit Mihail Megan, in which this issue is studied including in terms of applications in theory control.

Definitions and notations

We consider

- $(\Omega, \mathcal{B}, \mu)$ a probability space
- X a complex or real Banach spaces
- $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X
- $\|\cdot\|$ the norms on X and $\mathcal{B}(X)$ respectively
- I the identity operator on X
- $\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}$
- $T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$
- $L(\Omega, X, \mu)$ the Banach space of all Bochner measurable functions $f : \Omega \to X$ cu $\displaystyle\int_\Omega \|f(\omega)\| d\mu(\omega) < \infty$

Definitions and notations

Definition 2.1

A measurable random field $\varphi : \Delta \times \Omega \to \Omega$ is said to be a *stochastic evolution semiflow* on Ω if the following properties hold:

- (es_1) $\quad \varphi(t, t, \omega) = \omega$, for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$,
- (es_2) $\quad \varphi(t, s, \varphi(s, t_0, \omega)) = \varphi(t, t_0, \omega)$, for all $t \geq s \geq t_0 \geq 0$ and all $\omega \in \Omega$.

Definition 2.2

Let $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ be a measurable map. We say that Φ is a *stochastic evolution cocycle* associated to the stochastic evolution semiflow $\varphi : \Delta \times \Omega \to \Omega$ if the following conditions hold:

- (ec_1) $\quad \Phi(t, t, \omega) = I$ (the identity operator on X), for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$,
- (ec_2) $\quad \Phi(t, s, \varphi(s, t_0, \omega))\Phi(s, t_0, \omega) = \Phi(t, t_0, \omega)$, for all $t \geq s \geq t_0 \geq 0$ and all $\omega \in \Omega$.

If Φ represents a stochastic evolution cocycle over a stochastic evolution semiflow φ, then the pair $C = (\Phi, \varphi)$ is referred to as a *stochastic skew-evolution semiflow.*

Definitions and notations

Definition 2.3

The stochastic evolution cocycle $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ is said to be *reversible* if for all $(t, s, \omega) \in \Delta \times \Omega$, the map $\Phi(t, s, \omega)$ is bijective.

We denote by $\Phi(t, s, \omega) = \Phi^{-1}(s, t, \omega)$.

Definition 2.4

A mapping $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ with the property $P^2(s, \omega) = P(s, \omega)$ for all $(s, \omega) \in \mathbb{R}_+ \times \Omega$ is called *projections family* on X.

Remark 2.1

If $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is a projections family, then the mapping $Q : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ define as $Q(s, \omega) = I - P(s, \omega)$ also forms a projections family. This is referred to as the complementary projections family of P.

Definitions and notations

Definition 2.5

A projections family $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is said to be *invariant* to $C = (\Phi, \varphi)$ if

$$\Phi(t,s,\omega)P(s,\omega) = P(t,\varphi(t,s,\omega))\Phi(t,s,\omega),$$

for all $(t,s,\omega) \in \Delta \times \Omega$.

Remark 2.2

If P remains invariant for $C = (\Phi, \varphi)$, we denote by

$$\Phi_P(t,s,\omega) : \Delta \times \Omega \to \mathcal{B}(X)$$

the mapping defined by $\Phi_P(t,s,\omega) = \begin{cases} \Phi(t,s,\omega)P(s,\omega); \ t \geq s \\ \\ \Phi^{-1}(s,t,\omega)P(t,\varphi(t,s,\omega)); \ t < s. \end{cases}$,

for all $(t,s,\omega) \in \Delta \times \Omega$.

Definitions and notations

Proposition 2.1

If the stochastic evolution cocycle $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ *is reversible and the projection family* P *is invariant for the stochastic skew-evolution semiflow* $C = (\Phi, \varphi)$ *then* $P(s,\omega)\Phi^{-1}(t,s,\omega) = \Phi^{-1}(t,s,\omega)P(t,\varphi(t,s,\omega))$*, for all* $(t,s,\omega) \in \Delta \times \Omega$*.*

Proposition 2.2

If $\Phi_P(t,s,\omega) : \Delta \times \Omega \to \mathcal{B}(X)$ *and* $\Phi_P^{-1}(t,s,\omega)$ *is its inverse, then:*

(i) $\Phi(t,s,\omega)\Phi^{-1}(t,s,\omega)P(t,\varphi(t,s,\omega)) = P(t,\varphi(t,s,\omega))$, *for all* $(t,s,\omega) \in \Delta \times \Omega$*;*

(ii) $\Phi^{-1}(t,s,\omega)\Phi(t,s,\omega)P(s,\omega) = P(s,\omega)$, *for all* $(t,s,\omega) \in \Delta \times \Omega$*;*

(iii) $\Phi^{-1}(t,s,\omega)P(t,\varphi(t,s,\omega)) = P(s,\omega)\Phi^{-1}(t,s,\omega)P(t,\varphi(t,s,\omega))$, *for all* $(t,s,\omega) \in \Delta \times \Omega$*;*

Uniform h-dichotomy in mean

Definition 3.1

A nondecreasing map $h : \mathbb{R}_+ \to [1, \infty)$ with $\lim_{t \to \infty} h(t) = \infty$ is called a *growth rate.*

Definition 3.2

[25] The pair (C, P) is said to be *uniformly h-dichotomic in mean (u.h.d.m.)* if there are some constants $N > 1$ and $\nu > 0$ such that

$$(uhd_1m) \quad h(t)^\nu \int_\Omega \|\Phi(t, t_0, \omega)P(t_0, \omega)x_0(\omega)\| d\mu(\omega) \leq$$
$$Nh(s)^\nu \int_\Omega \|\Phi(s, t_0, \omega)P(t_0, \omega)x_0(\omega)\| d\mu(\omega);$$

$$(uhd_2m) \quad h(t)^\nu \int_\Omega \|\Phi(s, t_0, \omega))Q(t_0, \omega)x_0(\omega)\| d\mu(\omega) \leq$$
$$Nh(s)^\nu \int_\Omega \|\Phi(t, t_0, \omega))P(t_0, \omega)x_0(\omega)\| d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

If $h(t) = e^t$ and $h(t) = t + 1$, we infer the concepts of *u.e.d.m.* and *u.p.d.m.*

Uniform h-dichotomy in mean

Remark 3.1

The pair (C,P) is uniformly h-dichotomic in mean if and only if there exist $N>1$ and $\nu>0$ with

$$(uhd_1'm)\quad h(t)^{\nu}\int_{\Omega}\|\Phi(t,s,\omega)P(s,\omega)x(\omega)\|d\mu(\omega)\leq Nh(s)^{\nu}\int_{\Omega}\|P(s,\omega)x(\omega)\|d\mu(\omega);$$

$$(uhd_2'm)\quad h(t)^{\nu}\int_{\Omega}\|Q(s,\omega)x(\omega)\|d\mu(\omega)\leq Nh(s)^{\nu}\int_{\Omega}\|\Phi(t,s,\omega)Q(s,\omega)x(\omega)\|d\mu(\omega),$$

for all $(t,s,\omega)\in\Delta\times\Omega$ and $x\in L(\Omega,X,\mu)$.

Uniform h-dichotomy in mean

Theorem 3.1

The pair (C, P) is uniformly h-dichotomic in mean with Φ reversible stochastic evolution cocycle (i.e. $\Psi(t,s,\omega) = \Phi^{-1}(t,s,\omega)$) if and only if there are $N > 1$ and $\nu > 0$ with:

$$(uhd_1'''m)\quad h(t)^\nu \int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\| d\mu(\omega) \leq Nh(s)^\nu \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\| d\mu(\omega);$$

$$(uhd_2'''m)\quad h(t)^\nu \int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\| d\mu(\omega) \leq Nh(s)^\nu \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\| d\mu(\omega),$$

for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

Uniform h-growth in mean

Definition 4.1

[25] The pair (C, P) is said to be with *uniform h-growth in mean (u.h.g.m.)* if there exist constants $M \geq 1$ and $\alpha > 0$ such that:

$$(uhg_1m) \quad h(s)^\alpha \int_\Omega \|\Phi(t,t_0,\omega)P(t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq$$
$$Mh(t)^\alpha \int_\Omega \|\Phi(s,t_0,\omega)P(t_0,\omega)x_0(\omega)\|d\mu(\omega);$$

$$(uhg_2m) \quad h(s)^\alpha \int_\Omega \|\Phi(s,t_0,\omega))Q(t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq$$
$$Mh(t)^\alpha \int_\Omega \|\Phi(t,t_0,\omega))P(t_0,\omega)x_0(\omega)\|d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

As specific cases we note that when the growth rate is e^t, this establishes the concept of *uniform exponential growth in mean* and if the growth rate is $t + 1$, then we arrive at the concept of *uniform polynomial growth in mean* respectively.

Uniform h-growth in mean

Remark 4.1

The pair (C,P) has uniform h-growth in mean if and only if there exist $M > 1$ and $\alpha > 0$ with

$$(uhg_1'm) \quad h(s)^\alpha \int_\Omega \|\Phi(t,s,\omega)P(s,\omega)x(\omega)\| d\mu(\omega) \leq$$
$$Mh(t)^\alpha \int_\Omega \|P(s,\omega)x(\omega)\| d\mu(\omega);$$

$$(uhg_2'm) \quad h(s)^\alpha \int_\Omega \|Q(s,\omega)x(\omega)\| d\mu(\omega) \leq$$
$$Mh(t)^\alpha \int_\Omega \|\Phi(t,s,\omega)Q(s,\omega)x(\omega)\| d\mu(\omega),$$

for all $(t,s,\omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

Uniform h-growth in mean

Theorem 4.1

The pair (C, P) is uniformly h-dichotomic in mean with Φ reversible stochastic evolution cocycle (i.e. $\Psi(t, s, \omega) = \Phi^{-1}(t, s, \omega)$) if and only if there exist $M > 1$ and $\alpha > 0$ with:

$$(uhg_1'''m) \quad h(s)^\alpha \int_\Omega \|\Phi^{-1}(s, t_0, \omega) P(s, \varphi(s, t_0, \omega)) x_0(\omega)\| d\mu(\omega) \leq M h(t)^\alpha \int_\Omega \|\Phi^{-1}(t, t_0, \omega) P(t, \varphi(t, t_0, \omega)) x_0(\omega)\| d\mu(\omega);$$

$$(uhg_2'''m) \quad h(s)^\alpha \int_\Omega \|\Phi^{-1}(t, t_0, \omega) Q(t, \varphi(t, t_0, \omega)) x_0(\omega)\| d\mu(\omega) \leq M h(t)^\alpha \int_\Omega \|\Phi^{-1}(s, t_0, \omega) Q(s, \varphi(s, t_0, \omega)) x_0(\omega)\| d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

Integral characterizations for uniform h-dichotomy in mean

Definition 5.1

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow. We say that C is *strongly measurable* if, for all $(t_0, x) \in \mathbb{R}_+ \times L(\Omega, X, \mu)$, the mapping $s \mapsto \int_\Omega \|\Phi(s, t_0, \omega)x_0(\omega)\| d\mu(\omega)$, is measurable on $[t_0, \infty)$.

We denote by $\mathcal{H}$ the set of all functions $h : \mathbb{R}_+ \to [1, \infty)$ with the following properties:

- there exists $H > 1$ satisfying $h(t+1) \leq Hh(t), \forall\, t \geq 0$.
- for all $\beta < 0$ there exists $H_1 > 1$ with $\int\limits_s^\infty h(t)^\beta dt \leq H_1\, h(s)^\beta, \forall\, s \geq 0$.
- for all $\beta > 0$ there exists $alpha > 1$ with $\int\limits_0^t h(s)^\beta ds \leq alpha\, h(t)^\beta, \forall\, t \geq 0$.

Remark 5.1

If $h(t) = e^t$, *then* $h \in \mathcal{H}$.

Integral characterizations for uniform h-dichotomy in mean

Theorem 5.1

We assume that $C = (\Phi, \varphi)$ is a strongly measurable stochastic skew-evolution semiflow, (C, P) with uniform h-growth in mean and $h \in \mathcal{H}$. The pair (C, P) is uniformly h-dichotomic in mean if and only if there exist constants $D \geq 1$ and $d \in [0, 1)$ such that

$$(uhD_1^1m) \quad \int_s^\infty \frac{h(t)^d}{\int_\Omega \|\Phi^{-1}(t, t_0, \omega)P(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega)} dt \leq \frac{D\, h(s)^d}{\int_\Omega \|\Phi^{-1}(s, t_0, \omega)P(t, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega)};$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$, with $\int_\Omega \|\Phi^{-1}(s, t_0, \omega)P(s, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega) \neq 0$;

$$(uhD_2^1m) \quad \int_s^\infty h(t)^d \left(\int_\Omega \|\Phi^{-1}(t, t_0, \omega)Q(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega) \right) dt \leq D\, h(s)^d \int_\Omega \|\Phi^{-1}(s, t_0, \omega)Q(s, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

Corollary 5.2

We suppose that $C = (\Phi, \varphi)$ is a strongly measurable stochastic skew-evolution semiflow, (C, P) with uniform exponential growth in mean. The pair (C, P) is uniformly exponentially dichotomic in mean if and only if there exist constants $D \geq 1$ and $d \in [0, 1)$ with

$$(ueD_1^1m) \quad \int_s^\infty \frac{e^{dt}}{\int_\Omega \|\Phi^{-1}(t, t_0, \omega)P(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega)} dt \leq \frac{D\, e^{ds}}{\int_\Omega \|\Phi^{-1}(s, t_0, \omega)P(s, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega)};$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$,
with $\int_\Omega \|\Phi^{-1}(s, t_0, \omega)Q(s, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega) \neq 0$;

$$(ueD_2^1m) \quad \int_s^\infty e^{dt} \left(\int_\Omega \|\Phi^{-1}(t, t_0, \omega)Q(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega) \right) dt \leq D\, e^{ds} \int_\Omega \|\Phi^{-1}(s, t_0, \omega)Q(s, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

Integral characterizations for uniform h-dichotomy in mean

Theorem 5.3

Consider $C = (\Phi, \varphi)$ as a strongly measurable stochastic skew-evolution semiflow., (C, P) has uniform h-growth in mean and $h \in \mathcal{H}$. The pair (C, P) is uniformly h-dichotomic in mean if and only if there exist constants $D \geq 1$ and $d \in [0, 1)$ such that

$$(uhD_1^2m) \quad \int_{t_0}^{t} \frac{\int_\Omega \|\Phi^{-1}(s, t_0, \omega)P(s, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega)}{h(s)^d} ds \leq \frac{D \int_\Omega \|\Phi^{-1}(t, t_0, \omega)P(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega)}{h(t)^d};$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

$$(uhD_2^2m) \quad \int_{t_0}^{t} \frac{h(s)^{-d}}{\int_\Omega \|\Phi^{-1}(s, t_0, \omega)Q(s, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega)} ds \leq \frac{D\, h(t)^{-d}}{\int_\Omega \|\Phi^{-1}(t, t_0, \omega)Q(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega)},$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$ with $\int_\Omega \|\Phi^{-1}(t, t_0, \omega)Q(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega) \neq 0$.

Corollary 5.4

Let $C = (\Phi, \varphi)$ be a strongly measurable stochastic skew-evolution semiflow., (C, P) has uniform exponential growth in mean. The pair (C, P) is uniformly exponentially dichotomic in mean if and only if there exist some constants $D \geq 1$ and $d \in [0, 1)$ such that

$$(ueD_1^2m) \quad \int_{t_0}^{t} \frac{\int_\Omega \|\Phi^{-1}(s, t_0, \omega)P(s, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega)}{e^{ds}} ds \leq \frac{D \int_\Omega \|\Phi^{-1}(t, t_0, \omega)P(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega)}{e^{dt}};$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

$$(ueD_2^2m) \quad \int_{t_0}^{t} \frac{e^{-ds}}{\int_\Omega \|\Phi^{-1}(s, t_0, \omega)Q(s, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega)} ds \leq \frac{D\, e^{-dt}}{\int_\Omega \|\Phi^{-1}(t, t_0, \omega)Q(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega)},$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$

with $\displaystyle\int_\Omega \|\Phi^{-1}(t, t_0, \omega)Q(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega) \neq 0.$

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Open problems

- The generalization of these results for the nonuniform case.
- The generalization of these results for trichotomy.

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Thank you for your attention!

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