Discovery in learning math-Formulas for multiple order derivatives of rational functions

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Abstract

In Gazeta Matematică Series A (2002), we gave explicit formulas of the numerators of the first and second derivative of the ratios of polynomials of any degree in $\mathbb{R}[X]$, expressed with respectively without the help of determinants of coefficients of these polynomials.
New in present article: a generalization of a formula above to derivatives of any order, a divisibility property of the terms of these numerators, extension to the complex framework, a conjecture on the derivative of a ratio of power series and another conjecture.
We can use them in learning by discovery.

Keywords: rational functions, derivative, determinant, polynomial, discovery in learning math, greatest common divisor.

Introduction

From my teaching experience and that of other teachers, I know that one of the most difficult aspects of learning mathematics in the classroom is motivation, especially intrinsic motivation. Thus, in [5: Mathematics Teaching Methodology] it is concluded that one of the most important modern ideas that are the basis of mathematics teaching-learning methodology is to present mathematics as an open, developing science."I [planned to] write so that the learner may always see the inner basis of the things he learns, even so that the source of invention appears, and therefore in such a way that the learner may understand all as if he invented it himself.", G. W. von Leibnitz: Mathematische Schriften, edited by Gerhardt, vol. VII, p. 9

Introduction

There are algorithms for computing higher-order derivatives of rational functions in $\mathbb{R}[X]$. But they involve either writing the rational function decomposed into simple fractions, and then recomposing the results, others depend on the value of the variable x, others are recursive algorithms that must be repeated for each argument x. The formulas below express the numerator of the derivatives after writing them as a rational fraction, to make it possible to find the zeros of the derivatives, their sign, etc.

(All demonstrations not done in this presentation are in the pdf document on the XGEN website.)

Preliminary section

Lema **1**: If $f, g : D \rightarrow R$, $t \in \mathbb{N}^*$ are some functions t times differentiable on $D = \mathbb{R} \setminus \{x_1, x_2, \dots, x_h\}$, $x_1, x_2, \dots, x_h$ being the possible zeros of the function g, then the derivative of the order t of the function $r : D \rightarrow \mathbb{R}$, $r(x) = \frac{f(x)}{g(x)}$ has, on D, the form:

$$r^{(t)}(x) = \frac{f^{[t]}(x)}{g^{t+1}(x)} \qquad (1)$$

where the function $f^{[t]}(x)$ from the numerator can be written as a determinant of the order $t+1$:

$$f^{[t]} = \begin{vmatrix} g & 0 & 0 & \cdots & 0 & 0 & f \\ g' & g & 0 & \cdots & 0 & 0 & f' \\ g'' & 2g' & g & \cdots & 0 & 0 & f'' \\ g''' & 3g'' & 3g' & g & 0 & 0 & f''' \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g^{(t-1)} & \binom{t-1}{1} g^{(t-2)} & \binom{t-1}{2} g^{(t-3)} & \cdots & \binom{t-1}{t-2} g^{(1)} & g & f^{(t-1)} \\ g^{(t)} & \binom{t}{1} g^{(t-1)} & \binom{t}{2} g^{(t-2)} & \cdots & \binom{t}{t-2} g^{(2)} & \binom{t}{t-1} g^{(1)} & f^{(t)} \end{vmatrix} \qquad (2)$$

with the elements $a_{ij} = \binom{i-1}{j-1} g^{(i-j)}$ for $t+1 \geq i \geq j \geq 1$ and zero in the rest, then replacing the last column with the successive derivatives of the function f, analogously as in the first column for g (for $t=1$ only the first and last columns appear).

Remark 1.

In the paper [3 Bourbaki], the *Lemma* 1 is given without proof and in a slightly different form than in the one we rediscovered it.

Successively interchanging t times the last column with each of those to its left, the formula given in [3] is obtained.

The first two derivatives of rational functions

- ***Theorem 1***. Let be a rational function r defined on on a union of intervals D, $r(x) = \frac{f(x)}{g(x)}$ with f și g polynomials (which have no common roots) and max(degree f, degree g) = $n \in \mathbb{N}^*$, with $f(x) = \sum_{i=0}^{n} a_i x^i$ și $g(x) = \sum_{i=0}^{n} b_i x^i$. To calculate the first derivative r' we first write the matrix of coefficients:

$$A_1 = \begin{bmatrix} a_n & a_{n-1} & \dots & a_2 & a_1 & a_0 \\ b_n & b_{n-1} & \dots & b_2 & b_1 & b_0 \end{bmatrix}$$

and denote by $|ji|$ the minors $\begin{vmatrix} a_j & a_i \\ b_j & b_i \end{vmatrix}$ of matrix A_1, formed by columns j respectively i, columns numbered from left to right from n to 0.

To calculate the second derivative r'' we define the matrix:

$$A_2 = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_n & 0 \\ b_0 & b_1 & b_2 & \dots & b_n & 0 \\ 0 & b_0 & b_1 & \dots & b_{n-1} & b_n \end{bmatrix}$$

and $|ijk|$ the minors formed by the columns i, j and k of matrix A_2, columns numbered from left to right from 0 to n+1.

The first derivative of rational functions

- ***Theorem 1***. (continuation)

Then, over the entire definition domain D of the function r, we have the formula:

$$r'(x) = \frac{f_n^{[1]}(x)}{g^2(x)} \text{ cu } f_n^{[1]}(x) = \sum_{m=0}^{2(n-1)} c_m x^m \text{ where } c_m = \sum_{i+j=m+1}^{i<j} (j-i) \begin{vmatrix} a_j & a_i \\ b_j & b_i \end{vmatrix} =$$

$$\sum_{i+j=m+1}^{j>i} (j-i)|ji| \qquad (3);$$

The second derivative of rational functions

Theorem 1. (continuation)

Let be a rational function r defined on an open set D, $r(x) = \frac{f(x)}{g(x)}$ with f şi g polynomials (which have no common roots) and max(degree f, degree g) $= n \in \mathbb{N}^*$, with $f(x) = \sum_{i=0}^{n} a_i x^i$ şi $g(x) = \sum_{i=0}^{n} b_i x^i$.

Then, over the entire domain of definition D of the function r, we have the formulas:

$r''(x) = \frac{f_n^{[2]}(x)}{g^3(x)}$ cu $f_n^{[2]}(x) = \sum_{m=0}^{3(n-1)} d_m x^m$, where d_m takes one of the equivalent forms:

$$d_m = \sum_{i+j+k=m+2} a_i b_j b_k (i-j)(i+j-2k-1) \quad (4) \Leftrightarrow$$

$$d_m = \sum_{i+j+k=m+2} \frac{a_i b_j b_k}{2} \{2i(i-1) + 6jk - (j+k)[2i + (j+k) - 1)]\} \quad (5) \Leftrightarrow$$

$d_m = \sum_{i=0}^{n} a_i \sum_{j+k=m+2-i}^{j<k} \{b_j b_k [2i(i-1) + 6jk - (j+k)(2i+j+k-1)] + \frac{1+(-1)^{m-i}}{2}(j-i)(j-i+1)b_j^2\}$, where the term $(j-i)(j-i+1)b_j^2$ with $2j=m+2-i$, appears at most for m-i even number (6) $\Leftrightarrow$

$d_m = \sum_{i+j+k=m+3}^{i<j<k} (j-i)(k-j)(k-i)|ijk|$ (7)

The first two derivatives of rational functions

Theorem 1. (continuation)

- A proof can be found in [1].
- In the article in Word format we explained the proofs step by step and also found another way of proving.

The first two derivatives of rational functions

Theorem 1. (continuation)

Explicit examples of derivation. First derivative formulas in terms of determinants are very elegant for *n and t* ≤ 2 (and beyond):

For t=1, (3) $\overset{n=1}{\Longrightarrow}$ for $r_1(x) = \frac{a_1x+a_0}{b_1x+b_0}$, we have $A_1 = \begin{bmatrix} a_1 & a_0 \\ b_1 & b_0 \end{bmatrix}$

and $r'_1(x) = \frac{|10|}{(b_1x+b_0)^2} = \frac{\begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix}}{(b_1x+b_0)^2} = \frac{\det A_1}{(b_0+b_1x)^2}$;

For example if $r_1(x) = \frac{5x-1}{-x+4}$, $A_1 = \begin{bmatrix} 5 & -1 \\ -1 & 4 \end{bmatrix} \overset{|10|=19}{\Longrightarrow} \quad r'_1(x) = \frac{19}{(-x+4)^2}$;

For t=1, (3) $\overset{n=2}{\Longrightarrow}$ $r_2(x)=\frac{a_2x^2+a_1x+a_0}{b_2x^2+b_1x+b_0}$ we have A_1 =

$\begin{bmatrix} a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 \end{bmatrix}$, and then $\Rightarrow$ $r'_2(x)=\frac{(|21|x^2+2|20|x+|10|)}{(b_2x^2+b_1x+b_0)^2}$.

For example if $r_2(x) = \frac{-2x^2+5x-1}{3x^2-x+4}$, avem $A_1 = \begin{bmatrix} -2 & 5 & -1 \\ 3 & -1 & 4 \end{bmatrix} \Rightarrow$

$r'_2(x) = \frac{-13x^2-10x+19}{(3x^2-x+4)^2}$.

The first two derivatives of rational functions

Theorem 1. (continuation)

***Explicit examples of derivation*: second derivative formulas in terms of determinants are also very elegant for *n and t* ≤ 2** (and beyond). For ***t*=2:** (7) $\overset{n=1}{\Longrightarrow} r_1(x) = \dfrac{a_0 + a_1 x}{b_0 + b_1 x}$ we have $A_2 = \begin{bmatrix} a0 & a1 & 0 \\ b0 & b1 & 0 \\ 0 & b0 & b1 \end{bmatrix}$ with

$|012| = \det A_2$ so $r''_1(x) = \dfrac{2 \det A_2}{(b_0+b_1 x)^3}$;

$n = 1$, $r_1(x) = \dfrac{-1+5x}{4 - x}$, $\Rightarrow A_2 = \begin{bmatrix} -1 & 5 & 0 \\ 4 & -1 & 0 \\ 0 & 4 & -1 \end{bmatrix}$ cu det A_2 = 19, $\Rightarrow r''_1(x) = \dfrac{2\cdot 19}{(4-x)^3} = \dfrac{38}{(4-x)^3}$;

n = 2: (7) $\overset{n=2}{\Longrightarrow} r_2(x) = \dfrac{a_0+a_1x+a_2x^2}{b_0+b_1x+b_2x^2}$, $A_2 = \begin{bmatrix} a0 & a1 & a2 & 0 \\ b0 & b1 & b2 & 0 \\ 0 & b0 & b1 & b2 \end{bmatrix} \Rightarrow$

n = 2 $\Rightarrow r_2''(x) = \dfrac{2(|012|+3|013|x+3|023|x^2+|123|x^3)}{(b_0+b_1 x)^3}$. For e.g.: $r_2(x) = \dfrac{-1+5x-2x^2}{4-x+3x^2}$, $\Rightarrow$

$A_2 = \begin{bmatrix} -1 & 5 & -2 & 0 \\ 4 & -1 & 3 & 0 \\ 0 & 4 & -1 & 3 \end{bmatrix}$ și $\Rightarrow |012| = \begin{vmatrix} -1 & 5 & -2 \\ 4 & -1 & 3 \\ 0 & 4 & -1 \end{vmatrix} = -1$, $|013| = \begin{vmatrix} -1 & 5 & 0 \\ 4 & -1 & 0 \\ 0 & 4 & 3 \end{vmatrix} = -57$,

$|023| = \begin{vmatrix} -1 & -2 & 0 \\ 4 & 3 & 0 \\ 0 & -1 & 3 \end{vmatrix} = 15$ şi $|123| = \begin{vmatrix} 5 & -2 & 0 \\ -1 & 3 & 0 \\ 4 & -1 & 3 \end{vmatrix} = 39 \Rightarrow r_2''(x) = \dfrac{2(-1-171x+45x^2+39x^3)}{(4-x+3x^2)^3}$.

Remark 2. The formulas (3) and (7) for $t = 1$ or $t = 2$, as well as for any t, (once demonstrated), show a symmetry (as in Newton's binomial) of the coefficients of the columns with respect to the column/columns in the middle of the matrix A_t.

Remark 3. With these easy-to-memorize formulas in which like monomials have been reduced, we can study general rational functions (with coefficients as parameters) from the point of view of derivatives, monotony, extrema, convexity, etc. For example, constructing the function r(x) so that the half of the free term of the numerator of its second derivative, i. e. the determinant |012|, to be zero, we obtain that that numerator will reduce to a trinomial of the 2nd degree and a factor of the first degree, so we can easily study its sign, etc.:

$$r''(x) = \left(\frac{3+2x+3{,}75x^2}{6+7x+11x^2}\right)'' = \frac{-22x(17x^2-126x-108)}{(11x^2+7x+6)^3}.$$

Derivatives of arbitrary order of the rational functions

- ***Theorem 2***.

The same as in *Theorem* 1, natural number coefficients or indices i i_0, i_1, ..., i_{t+1}, n, t, m, p, etc. either belong to the set M = {0, 1, ..., n}, or they are null and cancel out the monomials in power of x, respectively the coefficients they index. Then ∀ t ≥ 1, the numerator $f^{[t]}$ of formula (1) equals

$$f^{[t]} = \sum_{m=0}^{(t+1)(n-1)} h_m x^m, \text{ unde } h_m = \sum_{\substack{i_0, i_1, \ldots, i_t \in M \\ i_0+i_1+\cdots+i_t = m+t}} a_{i_0} b_{i_1} b_{i_2} \ldots b_{i_t} \cdot$$

$$\cdot (i_0 - 1 \cdot i_1 - 0)(i_0 + i_1 - 2i_2 - 1) \ldots (i_0 + i_1 + \cdots + i_{t-1} - t \cdot i_t - t + 1) \quad (8)$$

Remark 4 . From this theorem, for t = 2, we obtain a second demonstration of formula (4)

Derivatives of arbitrary order of the rational functions

- ***The greatest common divisor of the coefficients of the numerators*** $f^{[t]}$

Lemma 2. The product of t consecutive natural numbers is divisible by t factorial = $t!$.

A demonstration can be found for example in [2].

- We agree to call numerical coefficients of a polynomial the real numbers by which the symbolic coefficients of the polynomial have been multiplied. For example for $f(x) = 2a_2XY^2 - 3a_1XZ$ the numerical coefficients are 2 and -3.

***Lemma* 3**. The derivative of order $t \le n$ of any polynomial of degree n, with integer coefficients with the GCD (greatest common divisor) of its numerical coefficients = 1, is a polynomial of degree n-t whose numerical coefficients have GCD = $t!$. If $t \ge n+1$ then $f^{(t)}(x) \equiv 0$ so all coefficients equal $0 \vdots t!$.

Proof. For a polynomial $f(x) = \sum_{j=0}^{n} a_i x^i$ we have $f'(x) = \sum_{i=1}^{n} i a_i x^{i-1}$, $f''(x) = \sum_{i=2}^{n} i(i-1) a_i x^{i-2}$ and in general $f^{(t)}(x) = \sum_{i=t}^{n} i(i-1)\cdots(i-t+1) a_i x^{i-t}$, which can be immediately verified by complete mathematical induction after the natural number $t \ge 1$.

As the numerical coefficients of monomials in $a_i x^{i-t}$ of the derivative $f^{(t)}(x)$ are $i(i-1)\ldots(i-t+1)$, which are products of t consecutive natural numbers, from *Lema* 2 $\Rightarrow$ they are all divisible by $t!$.

And because the numerical coefficient of the term in x^0 is exactly $t!$, any common divisor of the numerical coefficients of the polynomial $f^{(t)}(x)$ is a divisor of $t!$, so the GCD of the numerical coefficients of $f^{(t)}(x)$ is exactly $t!$.

Derivatives of arbitrary order of the rational functions

- ***The greatest common divisor of the coefficients of the numerators*** $f^{[t]}$

Theorem 3. Given the function $r : D \to R$, $r(x)=\frac{f(x)}{g(x)}$, where f and g are polynomials of any degree, that have no common roots, with the GCD of its numerical coefficients = 1, and r is not reduced to a polynomial in x. We know we have $r^{(t)}(x)=\frac{f^{[t]}(x)}{g^{t+1}(x)}$. Then $\forall\ t \in N^*$, the GCD of the numerical coefficients of the polynomial $f^{[t]}$ is equal to $t!$.

- *The proof given in the associated Word document is quite ingenious, combining skills about determinants, polynomials, integer divisibility and forms of the complete mathematical induction method.*

Derivatives of arbitrary order of the rational functions

Conjecture 1. Derivatives of any order $t \in \mathbb{N}^*$ of the function r, $r(x) = \frac{\sum_{j=0}^{n} a_i x^i}{\sum_{i=0}^{n} b_i x^i}$ from *Theorem 1*, of the form $r^{(t)}(x) = \frac{f_n^{[t]}(x)}{g^{t+1}(x)}$, can be calculated analogously as in *Theorem 1* as follows: we consider the matrix with t+1 lines and n+t columns:

$$A_t = \begin{bmatrix} a_0 & a_1 & \cdots & a_n & 0 & 0 & \cdots & 0 \\ b_0 & b_1 & \cdots & b_n & 0 & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots & 0 & b_0 & b_1 & \cdots & b_n & 0 \\ 0 & \ldots & 0 & 0 & b_0 & b_1 & \cdots & b_n \end{bmatrix},$$

the columns being numbered from left to right from 0 to $n+t-1$; then the

numerator of the derivative of order t $r^{(t)}(x)$ is the polynomial

$$f_n^{[t]}(x) = \sum_{m=0}^{(t+1)(n-1)} h_m x^m \qquad \text{where}$$

$$h_m = \frac{(-1)^t}{(t-1)!(t-2)!\cdots 2!1!} \sum_{i_0+i_1+\cdots+i_t\ = m+t(t+1)/2}^{0\le i_0<i_1<\cdots<i_t\le n+t-1} \left[\prod_{i,j\in\{i_0,i_1,\cdots,i_t\}}^{i<j} (j-i)\right] |i_0\, i_1\ \cdots\ i_t|$$

and $|i_0\, i_1\ \cdots\ i_t|$ are the minors of order $t+1$ of matrix A_t.

Conjecture 2 – The derivative of the power series

Conjecture. In case $f(x) = \sum_{j=0}^{\infty} a_i x^i$ şi g(x) = $\sum_{i=0}^{\infty} b_i x^i$ are convergent power series for $x \in$ (-R, R) with R $\in \mathbb{R} \setminus \{0\}$, g not being convergent to 0, if in addition functions $\left(\frac{\sum_{j=0}^{n} a_i x^i}{\sum_{i=0}^{n} b_i x^i}\right)$ and $\left(\frac{\sum_{j=0}^{n} a_i x^i}{\sum_{i=0}^{n} b_i x^i}\right)'$ are uniformly convergent on (-r, r), with r ≠ 0, then

$$\lim_{n\to\infty} \left(\frac{\sum_{j=0}^{n} a_i x^i}{\sum_{i=0}^{n} b_i x^i}\right)' = \frac{(\sum_{j=0}^{\infty} a_i x^i)}{(\sum_{i=0}^{\infty} b_i x^i)}.$$

Considering the infinite matrix $A_{2,\infty}$ having an infinity of columns and as (infinite) lines the coefficients a_0, a_1, ..., a_i, ... respectively b_0, b_1, ..., b_i, ... formulas (3) suggest that we would have:

$$\lim_{n\to\infty} \left(\frac{\sum_{j=0}^{n} a_i x^i}{\sum_{i=0}^{n} b_i x^i}\right)' = \frac{(\sum_{j=0}^{\infty} a_i x^i)}{(\sum_{i=0}^{\infty} b_i x^i)} = \frac{\sum_{m=0}^{\infty} x^m \sum_{i+j=m+1}^{j>i} (j-i)|j\,i|}{(\sum_{j=0}^{\infty} b_j x^j)^2}.$$

Similar for series with complex coefficients defined on an open disk $B(0, r)$.

Derivatives of arbitrary order of rational functions within the set of complex numbers

***Theorem* 4**. All of the above calculations can be extended in the same form within the framework of complex analysis. For this, everywhere the real numbers $x \in \mathbb{R}$ will be replaced by complex numbers $z \in \mathbb{C}$, the considered polynomials will be from $\mathbb{C}[X]$ and the expression "functions f and g are t times derivable on D, with $t \in N^*$ and $D = \mathbb{R} \setminus \{x_1, x_2, \ldots, x_h\}$, $x_1, x_2, \ldots, x_h$ being all possible zeros of the function g" will be replaced by "functions f and g are holomorphic on domain D, where $D = \mathbb{C} \setminus \{z_1, z_2, \ldots, z_h\}$, $z_1, z_2, \ldots, z_h$ being all possible complex zeros of the function g".

Conclusions

Derivation is the key operation for the study of variation of processes modeled by mathematics, physics (speed, acceleration, etc.), and the other sciences, because the “book of nature” is written in the language of mathematics (Galileo Galilei).

Among the applications of rational functions is the approximation of classes of functions, more precise than with polynomials, namely with Padé rational functions, thus obtaining more precise approximations of derivatives of different orders of these functions.

The applications of these formulas are facilitated by the fact that there are efficient numerical methods for calculating the determinants.

Conclusions

In addition to Remarks 2 and 3 made inline the text, we have that the above economic formulas motivate students for logical thinking, to improve the algorithm itself for solving a set of seemingly disparate problems (different examples with derivation of order I, II, III), necessary thinking also in computer science. Students can also, finding examples of clear and readable rules, gain confidence in themselves.

All this can occasion in class (or in exams or competitions) a contest-game of, for example, deriving once (or two, etc.) (individually or in groups) examples of rational functions or finding isolated coefficients of the monomials of specified degree of the polynomial from the numerator of such derivatives, some students by the classical method and others by these formulas and their comparison.

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