Formulas for derivatives of multiple order of rational functions, with/without determinants of coefficients- Proofs of theorems

Laurian Colcer

The first two derivatives of rational functions

Proof of Theorem 1.

a) To prove (3) we present the calculation in detail:

$f_n^{[1]}(x) = f'(x)g(x) - f(x)g'(x) = (\sum_{j=1}^{n} ja_j x^{j-1})(\sum_{i=0}^{n} b_i x^i) - (\sum_{i=0}^{n} a_i x^i)(\sum_{i=1}^{n} jb_j x^{j-1}) = \sum_{m=0}^{2n-1} x^m \sum_{i+j=m+1} ja_j b_i - \sum_{m=0}^{2n-1} x^m \sum_{i+j=m+1} ja_i b_j = \sum_{m=0}^{2n-1} x^m \sum_{i+j=m+1} j(a_j b_i - a_i b_j) =$

$$\sum_{m=0}^{2n-1} x^m \sum_{i+j=m+1}(ia_i b_j - ja_i b_j) \quad (8)$$

We will reduce like terms that appear in each c_m, highlighting the $|ji|$ minors of the matrix A_1:

$c_m = \sum_{i+j=m+1}^{j>i} j|ji| + \sum_{i+j=m+1}^{j<i} j|ji| = \sum_{i+j=m+1}^{j>i} j|ji| - \sum_{i+j=m+1}^{j<i} j|ij|$ = (swapping notations for i and j) $= \sum_{i+j=m+1}^{j>i} j|ji| - \sum_{i+j=m+1}^{j>i} i|ji|$, so $c_m = \sum_{i+j=m+1}^{j>i}(j-i)|ji|$, Q.E.D.

b) In order to find the form of the numerator $f_n^{[2]}(x)$ of the second derivative of the rational function, $r''(x) = \frac{f_n^{[2]}(x)}{g^{t+1}(x)}$, we are using the formula $f_n^{[2]} =$

$$\begin{vmatrix} g & 0 & f \\ g' & g & f' \\ g'' & 2g' & f'' \end{vmatrix} \quad (9)$$

which can be proved either independently by derivation twice or as a special case for $t = 2$ of *Lemma* 1. All terms below a_i, b_j and b_k with indexes i, j respectively k which are negative or greater than n are zero and the terms with those coefficients are considered zero (regardless of the power to which the indeterminate is raised x).

We notice that from (9) $\Rightarrow$ the degree of the polynomial $f_n^{[2]}$ is no greater than $3n$-2. We have:

$$f''(x) = \sum_{i=0}^{n} i(i-1)a_i x^{i-2},\ g'(x) = \sum_{j=0}^{n} jb_j x^{j-1},\ g''(x) = \sum_{j=0}^{n} j(j-1)b_j x^{j-2}.$$

Using the linearity of the determinant in (9) as a function of columns, we develop it as a sum:

$$f_n^{[2]}(x) = \sum_{i,j,k\in M} \begin{vmatrix} b_j x^j & 0 & a_i x^i \\ jb_j x^{j-1} & b_k x^k & ia_i x^{i-1} \\ j(j-1)b_j x^{j-2} & 2kb_k x^{k-1} & i(i-1)a_i x^{i-2} \end{vmatrix} =$$

(giving a common factor across each of the columns, then factoring the powers of x over the lines)

$$= \sum_{i,j,k\in M} a_i b_j b_k x^{i+j+k-2} \begin{vmatrix} 1 & 0 & 1 \\ j & 1 & i \\ j(j-1) & 2k & i(i-1) \end{vmatrix} =$$

$$= \sum_{i,j,k\in M} a_i b_j b_k x^{i+j+k-2} (i-j)(i+j-2k-1).$$

But since the terms for which $i = j$ are zero, we have in the sum $i \neq j$, so $\max(i+j+k-2) = (n-1)+n+n-2 = 3n-3$, thus

making the notation $i+j+k-2 = m$, we have

$$f_n^{[2]}(x) = \sum_{m=0}^{3n-3} d_m x^m \text{ where } d_m =$$

$$= \sum_{i+j+k=m+2} a_i b_j b_k (i-j)(i+j-2k-1)$$

which proves the formula (4).

- We note that in the last sum we can interchange b_j with b_k, so this sum is symmetrical in j and k, so we can write

$$2d_m = \sum_{i+j+k=m+2} a_i b_j b_k [(i-j)(i+j-2k-1) + (i-k)(i+k-2j-1)] =$$

$$\sum_{i+j+k=m+2} a_i b_j b_k [2i^2 - 2i - 2i(j+k) + 4jk + (j+k) - j^2 - k^2] =$$

$$\sum_{i+j+k=m+2} a_i b_j b_k \{2i(i-1) + 6jk - (j+k)[2i + (j+k) - 1)]\} \quad \Rightarrow$$

$$d_m = \sum_{i+j+k=m+2} \frac{a_i b_j b_k}{2} \{2i(i-1) + 6jk - (j+k)[2i + (j+k) - 1)]\} \Rightarrow \text{(5) holds.}$$

To demonstrate (6), we write that (5) $\Rightarrow$ $2d_m = \sum_{i=0}^{n} a_i \sum_{j+k=m+2-i} B_{ijk}$,

where we note $B_{ijk} \stackrel{\text{def}}{=} \{2i(i-1) + 6jk - (j+k)[2i + (j+k) - 1)]\} b_j b_k$.

Now since the B_{ijk} terms are obviously symmetric in j and k, we deduce that

$$\sum_{j+k=m+2-i} B_{ijk} = \sum_{j+k=m+2-i}^{j<k} B_{ijk} + \sum_{j+k=m+2-i}^{j=k} B_{ijk} + \sum_{j+k=m+2-i}^{j>k} B_{ijk} =$$

$= 2\sum_{j+k=m+2-i}^{j<k} B_{ijk} + \sum_{j+k=m+2-i}^{j=k} B_{ijk} = 2\sum_{j+k=m+2-i}^{j<k} B_{ijk} + 2(j-i)(j-i+1)b_j^2$, where $2j = m+2-i$,

so the terms $2(j-i)(j-i+1)b_j^2$ appear at most for $m-i$ even; so

$$\tfrac{1}{2}\sum_{j+k=m+2-i} B_{ijk} = \sum_{j+k=m+2-i}^{j<k} B_{ijk} + \frac{1}{2}\sum_{j+k=m+2-i}^{j=k} B_{ijk} \quad \Rightarrow$$

$$d_m = \sum_{i=0}^{n} a_i \sum_{j+k=m+2-i}^{j<k} \{b_j b_k [2i(i-1) + 6jk - (j+k)(2i+j+k-1)] + \frac{1+(-1)^{m-i}}{2}(j-i)(j-i+1)b_j^2\},$$

so formula (6) is also true.

- It remains to demonstrate (7): we denote the presumptive numerator by $\sum_{m=0}^{3(n-1)} e_m x^m$ with $e_m =$

$\sum_{i+j+k=m+3}^{i<j<k} (j-i)(k-j)(k-i)|ijk|$ and we wish to prove that e_m is identical to d_m from formula (6).

Indeed, making the notation $|jk| = \begin{vmatrix} b_j & b_k \\ b_{j-1} & b_{k-1} \end{vmatrix}$, and developing the determinant $|ijk|$ after the first line we get $|ijk| = a_i|jk| - a_j|ik| + a_k|ij|$ so, noting the products $(j-i)(k-j)(k-i)$ with Π_{ijk}, we get

$$e_m = \sum_{i+j+k=m+3}^{i<j<k} \Pi_{ijk} a_i |jk| - \sum_{i+j+k=m+3}^{j<i<k} \Pi_{ijk} a_j |ik| + \sum_{i+j+k=m+3}^{j<k<i} \Pi_{ijk} a_k |ij|$$

then, interchanging in notation j with i in the second sum and bearing in mind that $\Pi_{jik} =$

- Π_{ijk} , and in the third sum making the circular permutation ($k \to i$, $i \to j$, $j \to k$) and since $\Pi_{kij} = \Pi_{ijk}$, we have

$e_m = \sum_{i+j+k=m+3}^{i<j<k} \Pi_{ijk} a_i\, |jk| + \sum_{i+j+k=m+3}^{j<i<k} \Pi_{ijk} a_i\, |jk| + \sum_{i+j+k=m+3}^{j<k<i} \Pi_{ijk} a_i |jk| =$

(the order in which the summation index i is placed relative to j and k is arbitrary)

$= \sum_{i+j+k=m+3}^{j<k} \Pi_{ijk}\, a_i\, |jk| =$

(grouping the terms in ai a_i and noting $s' = m+3-i$) $= \sum_{i=0}^{n} a_i \sum_{k+j=s'} D_{ijk}$, where

$$D_{ijk} \overset{\text{def}}{=} \sum_{k+j=s'}^{j<k} \Pi_{ijk}\, |jk| = \sum_{k+j=s'}^{j<k} \Pi_{ijk}\, (b_j b_{k-1} - b_{j-1} b_k)$$

$$= \sum_{k+j=s'}^{j<k} (j-i)(k-j)(k-i) b_j b_{k-1} - \sum_{k+j=s'}^{j\le k} (j-i)(k-j)(k-i) b_{j-1} b_k;$$

in order to find and group together similar terms, let's examine the coefficients with which they appear in the algebraic sum, substituting in the first sum $k = k'+1$, respectively in the second sum $j = j'+1$. We have

$D_{ijk} = \sum_{i+k'+j=m+2-i}^{j\le k'} (j-i)(k'+1-j)(k'+1-i) b_j b_{k'} - \sum_{i+k+j'=m+2-i}^{j'<k} (j'+1-i)(k-j'-1)(k-i) b_{j'} b_k \Rightarrow$

(renoting j', k' with j respectively with k and factoring $b_j b_k$ to reduce the like terms two by two) $\Rightarrow D_{ijk} = \sum_{j+k=m+2-i}^{j<k} [(j-i)(k+1-j)(k+1-i) - (j+1-i)(k-j-1)(k-i)] b_j b_k$

$+(j-i)(j-i+1)\boldsymbol{b_j^2}$ where $2j=m+2-i$, that is, the term in $\boldsymbol{b_j^2}$ appears at most for $m-i$ even number and, doing the calculation in parentheses, we get: $D_{ijk} =$ $\sum_{j+k=m+2-i}^{j<k} \{\boldsymbol{b_j b_k}[2i(i-1)+6jk - (j+k)(2i+j+k-1] + \frac{1+(-1)^{m-i}}{2}(j-i)(j-i+1)\boldsymbol{b_j^2}\}$, so $e_m = d_m$ from the proven formula (6), so equality (7) is also true, Q.E.D.

Derivatives of arbitrary order of the rational functions

Proof of Theorem 2.

We will use the method of complete mathematical induction.

For t = 1 respectively, we proved formula (9).

We will deduce the formula of $f_n^{[t+1]}(x)$ from that of $f_n^{[t]}(x)$ for any $t \ge 1$ (8). We have:

$$f_n^{[t+1]}(x) = (f_n^{[t]})'(x)g(x) - (t+1)f_n^{[t]}(x)g'(x). \qquad (9)$$

We note $(i_0 - 1 \cdot i_1 - 0)(i_0 + i_1 - 2i_2 - 1) \dots (i_0 + i_1 + \dots + i_{t-1} - t \cdot i_t - t + 1)$ with $p(t)$.

We will calculate the two terms of difference (9) in turn:

$(f_n^{[t]})'(x)g(x) = (\sum_{m=0}^{(t+1)n-t-1} h_m x^m)' g(x) = (\sum_{m=0}^{(t+1)n-t-2} x^m (m+1) h_{m+1})(\sum_{i_{t+1}=0}^{n} b_{i_{t+1}} x^{i_{t+1}})$ =

$$= \sum_{p=0}^{(t+2)(n-1)} x^p \sum_{m+i_{t+1}=p} (m+1) b_{i_{t+1}} h_{m+1} =$$

$$= \sum_{p=0}^{(t+2)(n-1)} x^p \sum_{m+i_{t+1}=p} (m+1) b_{i_{t+1}} \sum_{i_0+i_1+\dots+i_t=(m+1)+t}^{i_0,i_1,\dots,i_t \in N} a_{i_0} b_{i_1} b_{i_2} \dots b_{i_t} p(t) =$$

$$= \sum_{p=0}^{(t+2)(n-1)} x^p \sum_{i_0+i_1+\dots+i_t-t-1+i_{t+1}=p} (i_0 + i_1 + \dots + i_t - t)\, a_{i_0} b_{i_1} b_{i_2} \dots b_{i_t} b_{i_{t+1}} p(t) =$$

$$\sum_{p=0}^{(t+2)(n-1)} x^p \sum_{i_0+i_1+\cdots+i_t+i_{t+1}=p+t+1} a_{i_0} b_{i_1} b_{i_2} \dots b_{i_t} b_{i_{t+1}} p(t)(i_0+i_1+\cdots+i_t-t) \qquad (10)$$

To find out the second term in (9), we calculate

$$(t+1) f_n^{[t]}(x) g'(x) = (t+1)\left(\sum_{m=0}^{(t+1)n-t-1} h_m x^m\right)\left(\sum_{i_{t+1}=0}^{n-1} (i'_{t+1}+1) b_{i'_{t+1}+1} x^{i'_{t+1}}\right) =$$

$$=(t+1)\sum_{p=0}^{(t+2)(n-1)} x^p \sum_{m+i_{t+1}=p} (i'_{t+1}+1) b_{i'_{t+1}+1} \sum_{i_0+i_1+\cdots+i_t=(m+1)+t}^{i_0,i_1,\dots,i_t \in N} a_{i_0} b_{i_1} b_{i_2} \dots b_{i_t} p(t) =$$

(Noting $i'_{t+1}+1 = i_{t+1}$)

$$= \sum_{p=0}^{(t+2)(n-1)} x^p \sum_{m+i_{t+1}=p} (t+1) i_{t+1} b_{i_{t+1}} \sum_{i_0+i_1+\cdots+i_t=(m+1)+t}^{i_0,i_1,\dots,i_t \in N} a_{i_0} b_{i_1} b_{i_2} \dots b_{i_t} p(t) =$$

$$\sum_{p=0}^{(t+2)(n-1)} x^p \sum_{i_0+i_1+\cdots+i_t+i_{t+1}=p+t+1} a_{i_0} b_{i_1} b_{i_2} \dots b_{i_t} b_{i_{t+1}} p(t)(t+1) i_{t+1}$$

From (9) we get: $f_n^{[t+1]}(x) =$

$$\sum_{p=0}^{(t+2)(n-1)} x^p \sum_{i_0+i_1+\cdots+i_t+i_{t+1}=p+t+1} a_{i_0} b_{i_1} b_{i_2} \dots b_{i_t} b_{i_{t+1}} p(t)[i_0+i_1+\cdots+i_t-t-(t+1)i_{t+1}]$$

Proof of Theorem 3.

We first note the proposition "GCD of the numerical coefficients of the polynomial $f^{[t]}$ is equal to $t!$" cu P_t. We will prove it by complete mathematical induction after t. P_1 is obvious true from the formula for the derivation of a ratio of functions.

We now assume that for some positive integer t all P_1, P_2, ..., P_t are true. We shall then demonstrate that P_{t+1} is also true.

Lemma 3 tells us the formula (2) for $f^{[t]}$, $\forall\, t \in \mathbb{N}^*$, a non-identical null polynomial (11)
from the hypothesis, otherwise $r^{(t)}(x) \equiv 0$ and $r(x)$ would be a polynomial in x
so applying *Lemma* 1 to the t+1 order derivative, we have

$$f^{[t+1]} = \begin{vmatrix} g & 0 & 0 & \cdots & 0 & 0 & f \\ g' & g & 0 & \cdots & 0 & 0 & f' \\ g'' & 2g' & g & \cdots & 0 & 0 & f'' \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g^{(t-1)} & \binom{t-1}{1} g^{(t-2)} & \binom{t-1}{2} g^{(t-3)} & \cdots & g & 0 & f^{(t-1)} \\ g^{(t)} & \binom{t}{1} g^{(t-1)} & \binom{t}{2} g^{(t-2)} & \cdots & \binom{t}{t-1} g^{(1)} & g & f^{(t)} \\ g^{(t+1)} & \binom{t+1}{1} g^{(t)} & \binom{t+1}{2} g^{(t-1)} & \cdots & \binom{t+1}{t-1} g^{(2)} & \binom{t+1}{t} g^{(1)} & f^{(t+1)} \end{vmatrix} \qquad (12)$$

We develop the determinant (10) after the penultimate column and get:

$$f^{[t+1]} = -\binom{t+1}{1} g^{(1)} f^{[t]} + g \begin{vmatrix} g & 0 & \ldots & 0 & 0 & f \\ g' & g & \ldots & 0 & 0 & f' \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ g^{(t-2)} & \binom{t-2}{1} g^{(t-3)} & .. & g & 0 & f^{(t-2)} \\ g^{(t-1)} & \binom{t-1}{1} g^{(t-2)} & \ldots & \binom{t-1}{t-2} g^{(1)} & g & f^{(t-1)} \\ g^{(t+1)} & \binom{t+1}{1} g^{(t)} & \ldots & \binom{t+1}{t-2} g^{(3)} & \binom{t+1}{t-1} g^{(2)} & f^{(t+1)} \end{vmatrix},$$

Then we repeat the above procedure recurrently: we develop after the penultimate column the last determinant obtained (algebraic complement of element g) and finally we get:

$$f^{[t+1]} = [-\binom{t+1}{1} g^{(1)} f^{[t]} - \binom{t+1}{2} g^{(2)} f^{[t-1]} g - \binom{t+1}{3} g^{(3)} f^{[t-2]} g^2 - \ldots - \binom{t+1}{t+1} g^{(t+1)} f^{[0]} g^t\,] + f^{(t+1)} g^{t+1} =$$

$$-\sum_{k=1}^{t+1} \binom{t+1}{k} g^{(k)} f^{[t+1-k]} g^{k-1} + f^{(t+1)} g^{t+1}.$$

But because from the induction hypothesis and *Lemma 3* it follows that the numerical coefficients of any term polynomial term of the above sum are divisible by $\frac{(t+1)!}{k!(t+1-k)!} k!(t+1-k)! = (t+1)!$, We have that all the numerical coefficients of $f^{[t+1]}$ are divisible by $(t+1)!$. And from the induction hypothesis and from (11) the first term of the sum, the polynomial $\binom{t+1}{1} f^{[t]} g$ has the GCD of its numerical coefficients equal to $(t+1)!$, it follows that any common divisor of the numerical coefficients of the polynomial $f^{[t]}$ is a divisor of $(t+1)!$, so GCD of the numerical coefficients of $f^{[t]}$ is precisely $(t+1)!$; so P_{t+1} is true. It follows, by virtue of mathematical induction, that *Theorem 3* is true.